Game Theory and Applications

1– Equilibria

November 2012, Departamento de Ingeniería Industrial, Univ. of Santiago, Chile
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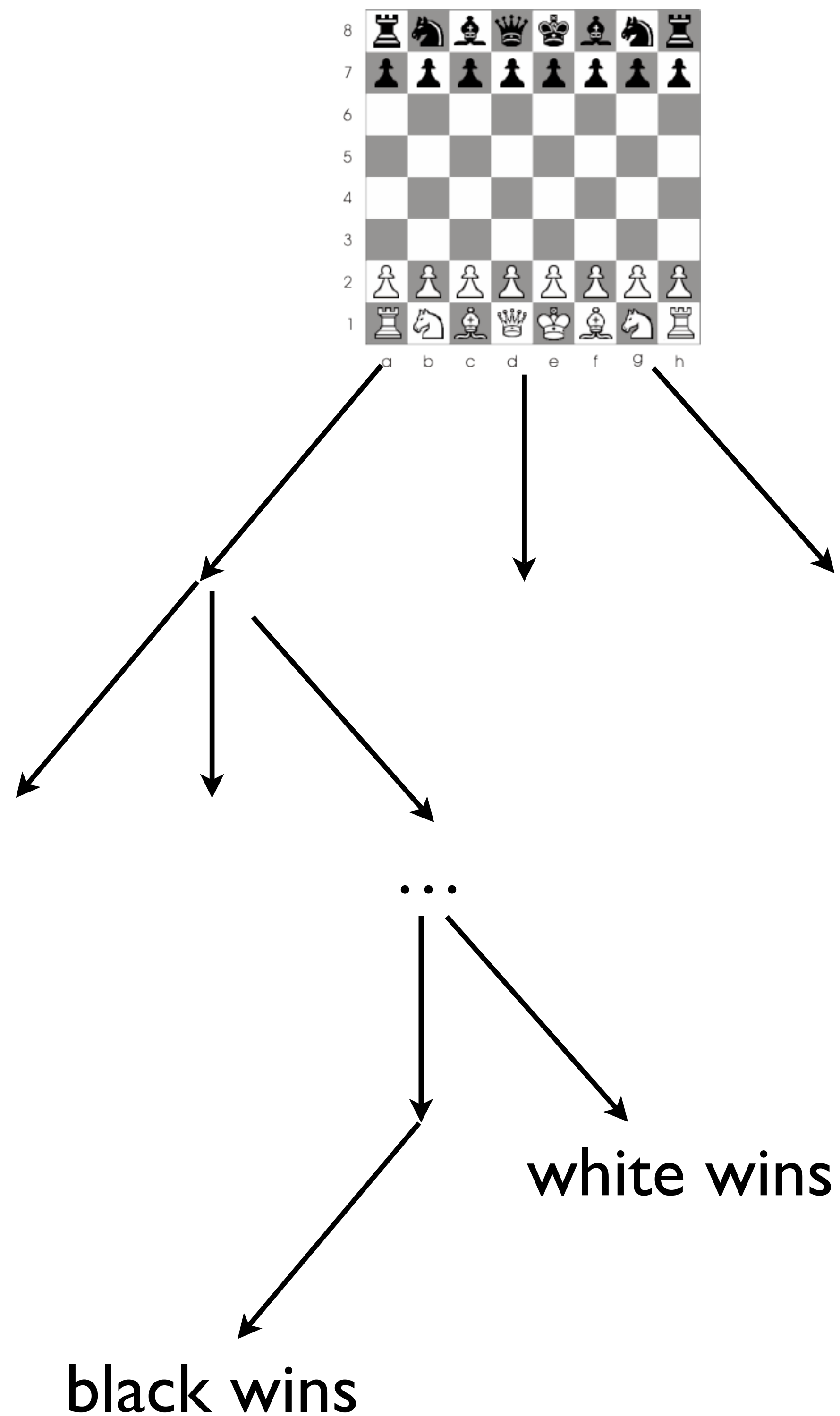
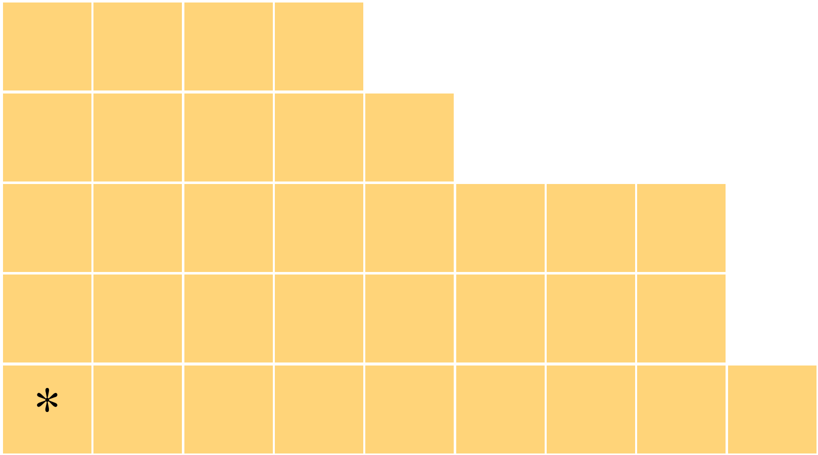
version 1

Repeated games

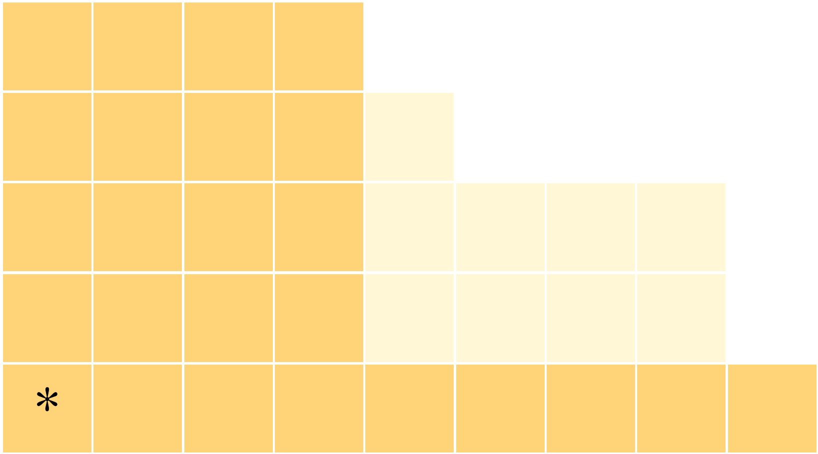
- two players white and black
- Players make actions in alternation
- White starts
- Except ties, either one of the player can always win
- In particular this is true for chess, except that computing the winning stragegy is out of scope



The chocolate game

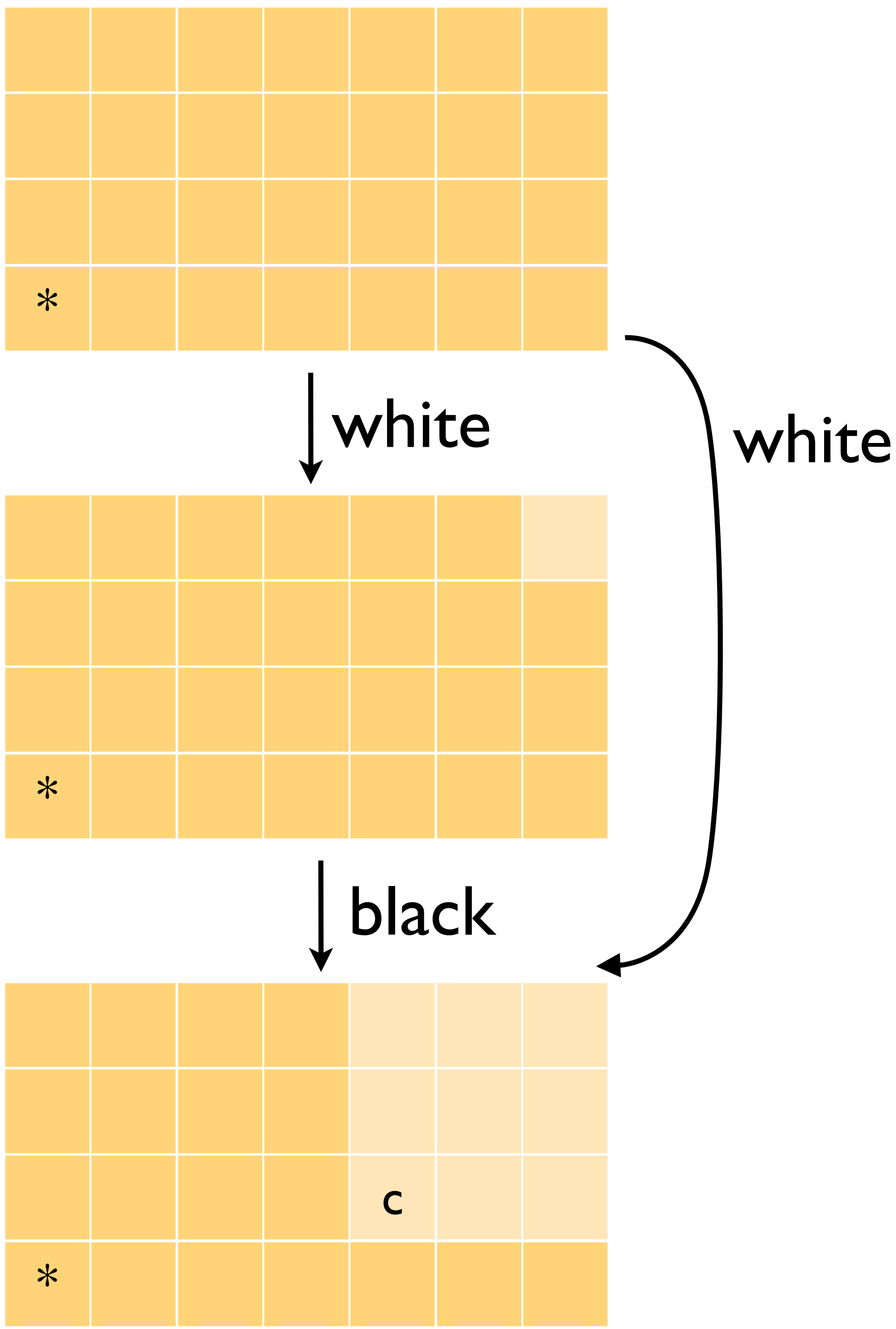


- we are given a tableau (chocolate bar)
- the lower-left cell is poisoned
- Every player has to select a cell and all cells above or to the right
- The last player to select the lower-left cell looses
- **Claim**: When the tableau is a rectangle larger than 1×1, white can win



Idea of the proof: change roles

- **For a proof by contradiction:** Suppose that black can always win for any rectangular initial configuration other than 1x1.
- Say white eats upper-right cell.
- And black eats some cell c such that he will win
- Now white could have played c in the first place and therefore win.



Strategic Games

- finite number of players
- each has to choose one out of a finite set of strategies
- chosen strategies form strategy profiles
- there is a payoff table u_i for each player mapping strategy profiles to a numerical utility
- player want to choose strategy that maximize their payoff
- pure Nash equilibria are strategy profiles s such that every player is happy, i.e.

$$\forall i: u_i(s) = \max_{s^*_i} u_i(s_{-i}, s^*_i)$$

notation: (s_{-i}, s^*_i) is the strategy profile obtained from s, when player i-th strategy is changed to s^*_i

the argmax s^*_i is called the **best response** for player i to s_{-i}

prisoner's dilemma

	don't confess	confess
don't confess	-1,-1	-4,0
confess	0,-4	-3,-3

no cooperative game

A couple goes to a concert

∃2 pure Nash equilibria

	Bach	Stravinsky
Bach	2,1	0,0
Stravinsky	0,0	1,2

	Bach	Stravinsky
Bach	2,1	0,0
Stravinsky	0,0	1,2

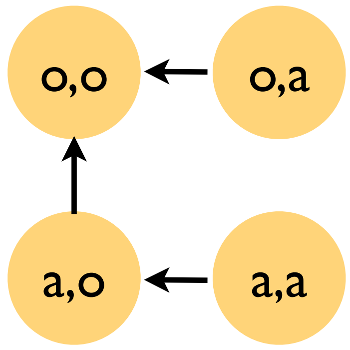
∃ single pure Nash equilibrium

	Mozart	Mahler
Mozart	2,2	0,0
Mahler	0,2	1,1

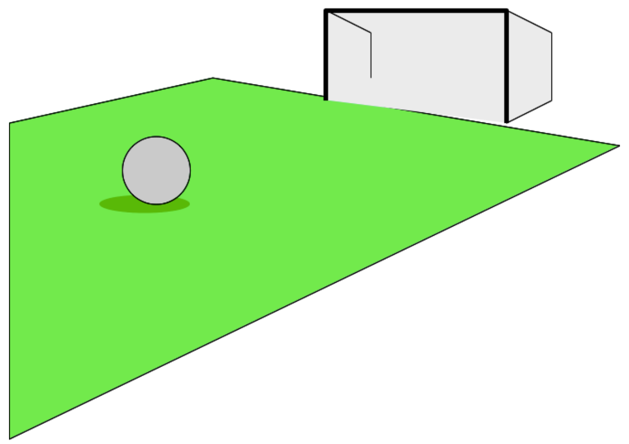
Best response dynamics

- Consider a directed graph
- Vertices are strategy profiles
- there is an arc from s to s' if $\exists i : s^*=(s_{-i},s^*_i)$ and $u_i(s^*)<u_i(s)$ and s^*_i is best response to s_{-i}
- s is a pure Nash equilibria iff it has out-degree 0

	Mozart	Mahler
o:= Mozart	2,2	0,0
a:= Mahler	0,2	1,1



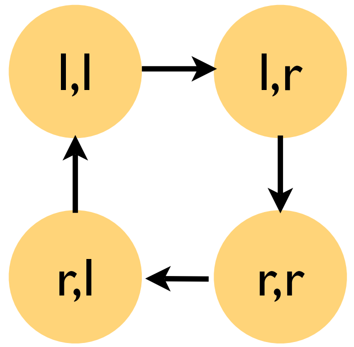
Penalty



- has no pure Nash equilibrium
- but has a **mixed Nash equilibrium**
- a mixed strategy X_i for a player is a distribution over his strategies
- a mixed strategy profile X is the product of the mixed strategies
- players want to maximize their expected payoff:

$$U_i(X)=\Sigma_s \Pr[X_i=s_i]\ u_i(s)$$

	left	right
left	1,-1	-1,1
right	-1,1	1,-1



mixed strategies are pure strategies in a larger game

- player i has finite strategy set S_i
- a mixed Nash equilibrium…
- **Fact**: pure Nash equilibria exist under some conditions A
- **Thm**: mixed Nash equilibria do always exist

- player i has infinite strategy set D_i, the set of probability distributions over S_i
- … corresponds to a pure Nash equilibrium
- these games satisfy conditions A

Nash’s theorem

- Sperners Lemma [?]

➡ Brouwer’s fixed point theorem [1915]

➡ Kakutani’s fixed point theorem [1941]

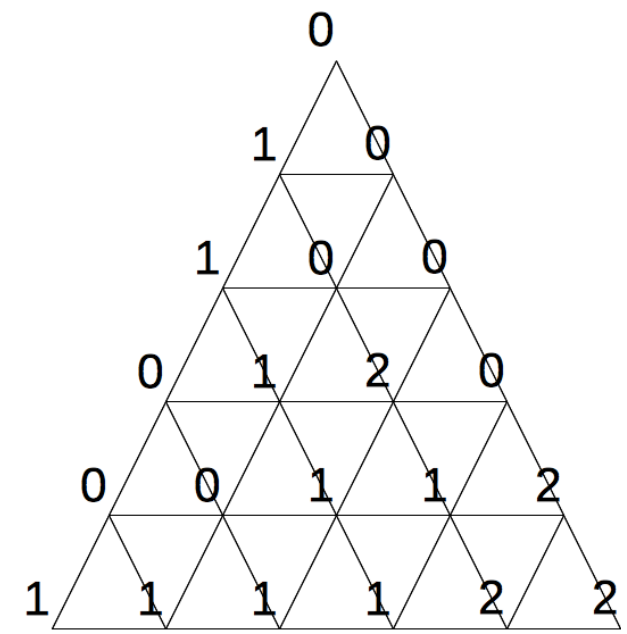
➡ Nash’s theorem: every finite game (finite players, finite strategies) has a mixed Nash equilibrium [1950]

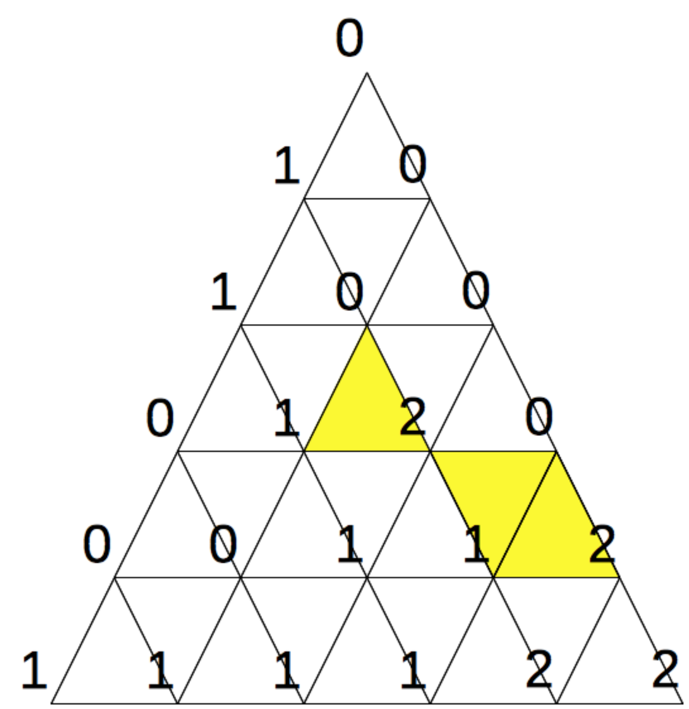
Sperner's lemma in 1 dimension

- Given a sequence of 0{0,1}*1, it has an odd number of 01 substrings
- 00010001001111000110111

Sperner's Lemma (in 2D)

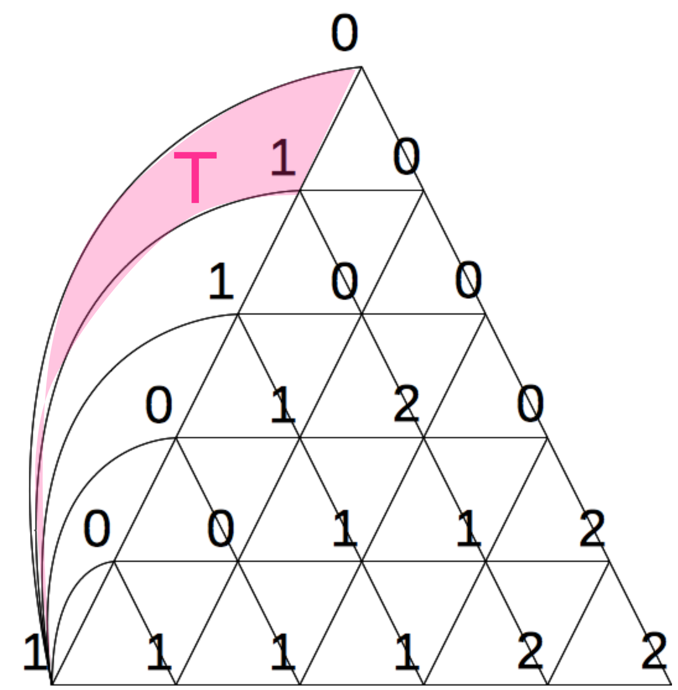
- Consider a triangulation of a triangle
- A {0,1,2}-labeling of vertices is **valid** if
  - the 3 corners have distinct labels
  - the vertices on the border are labeled with one of the adjacent corner labels
- **Sperner's Lemma:** in a valid labeling there is an odd number of {0,1,2}-labeled triangles

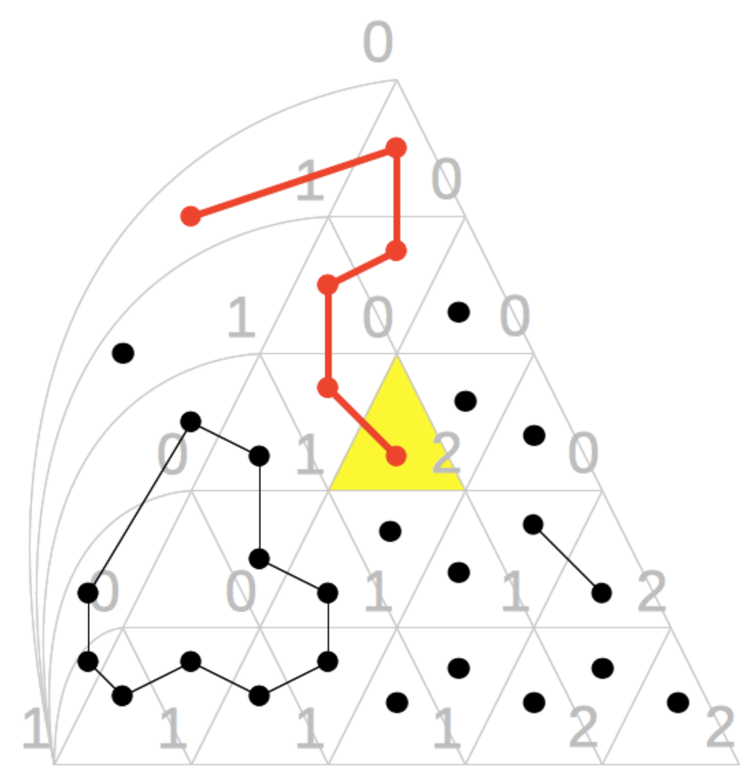




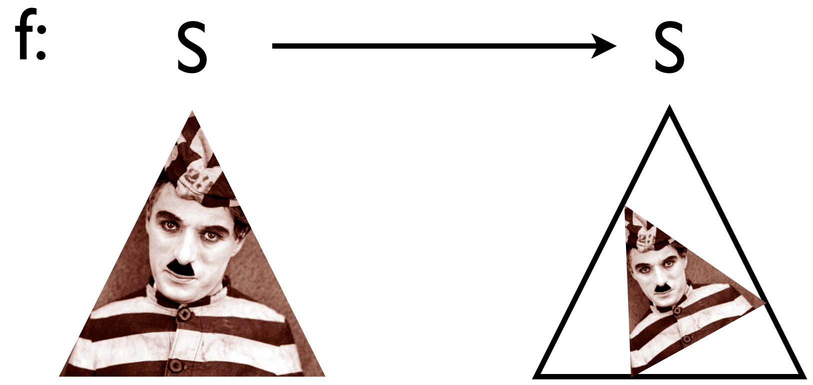
Constructive proof

- On the 01-side add triangles by connecting to the 1-corner. Let T be the outer triangle.
- Consider the graph, where triangles are vertices which are connected by an edge if they share a 01-side.
- In this graph every vertex has degree at most 2. 012-labeled triangles correspond to degree 1 vertices.
- The graph has an even number of degree 1 vertices. Since T is one of them, there is an odd number of 012-triangles. □
- To find one it suffices to follow the path starting at T.





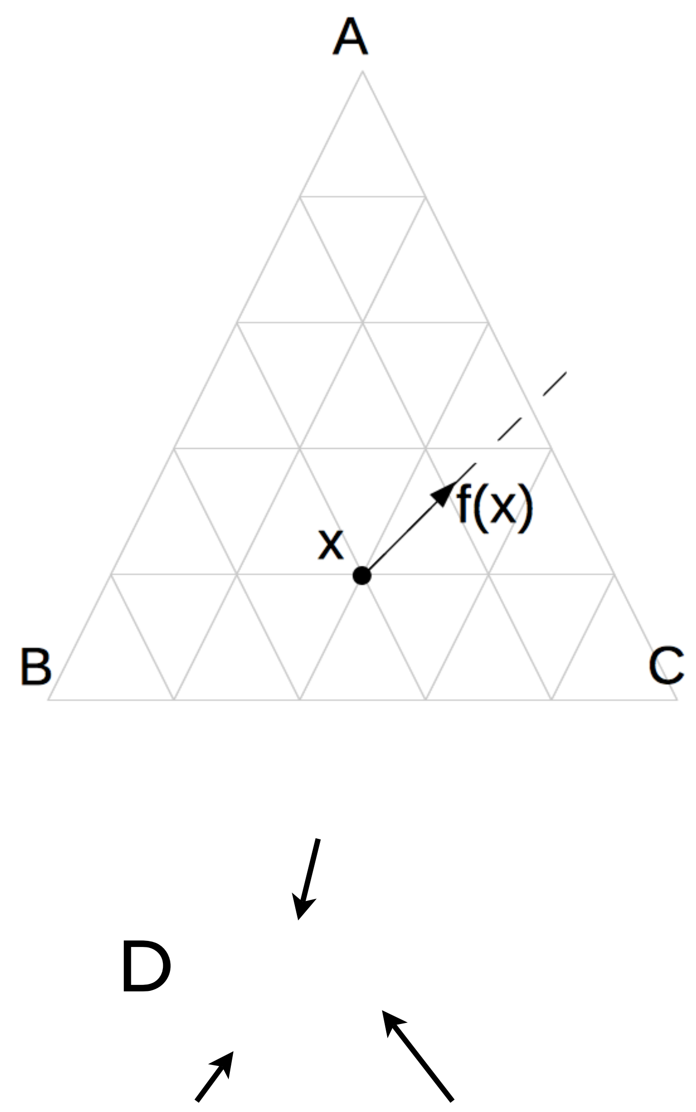
Brouwser's fixed point theorem



- Let f be any continuous function from a simplex S to S
- **Thm:** there is a fixpoint x∈S for f, i.e. f(x)=x

Idea of the proof: triangulate

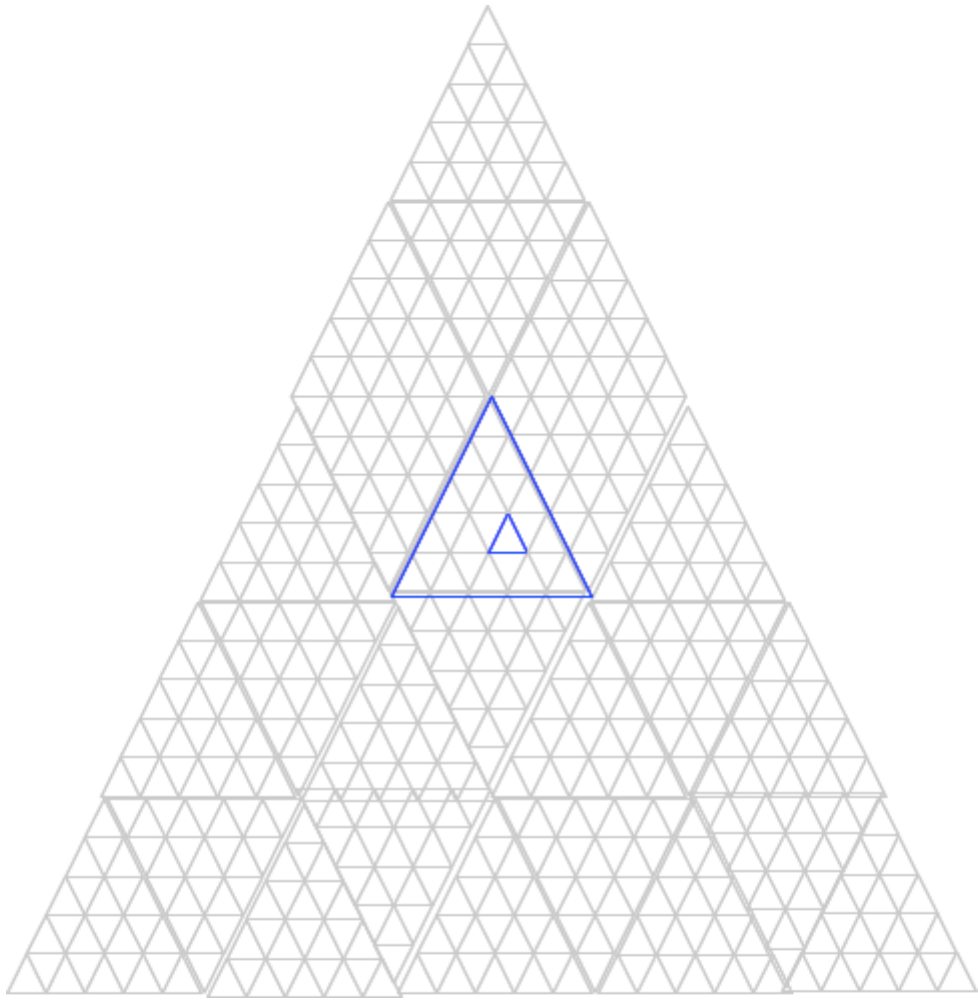
- Consider a triangulation T of S
- let A,B,C be 3 vertices of T on the border
- label vertex x of T with say 0 if the halfline starting at x in direction (x,f(x)) crosses the AC-border. (Similar for AB and BC-border)
- This is a valid labeling. So there is a 012-labeled triangle.
- (!) we do not necessarily have a fix point in D



Idea of the proof: refine

- Use finer and finer triangulations.
- As a result we have a sequence of 012-labeled triangles $D_0, D_1, \ldots$
- Let $x_0, x_1, \ldots$ be the centers of these triangles
- We can extract a subsequence that has a limit x^*
- Then since f is continuous, x^* is a fix point.

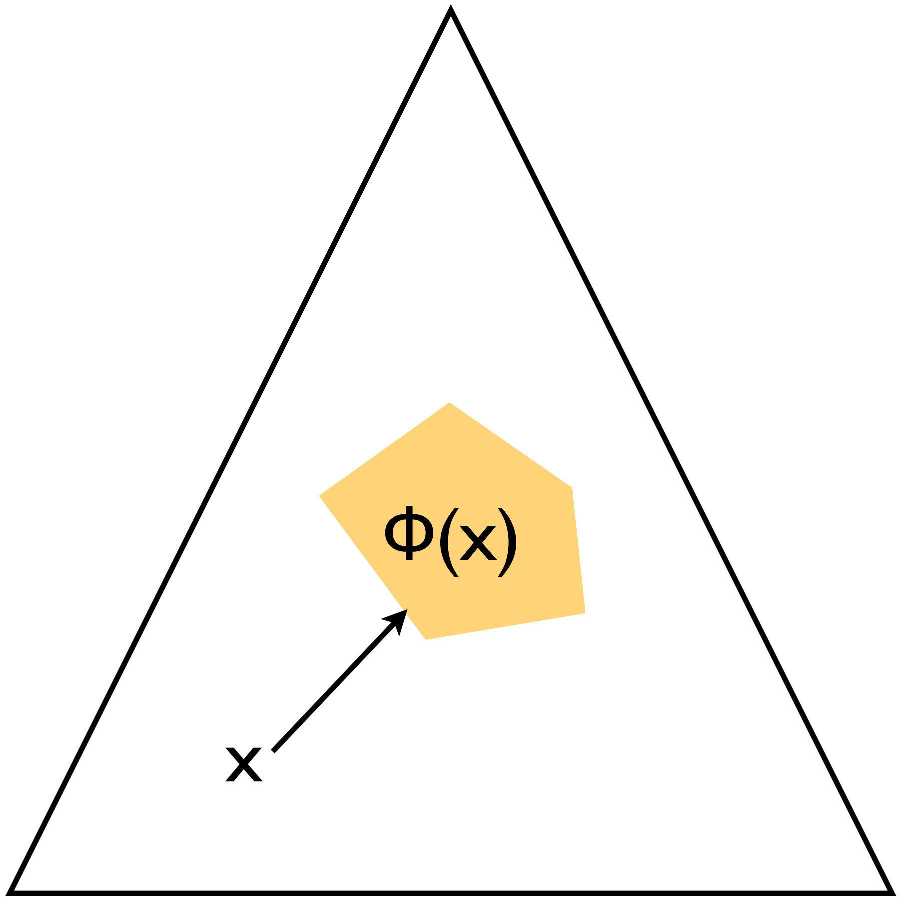
□



Kakutani's fixed point theorem

notation: 2^S consists of all subsets of S

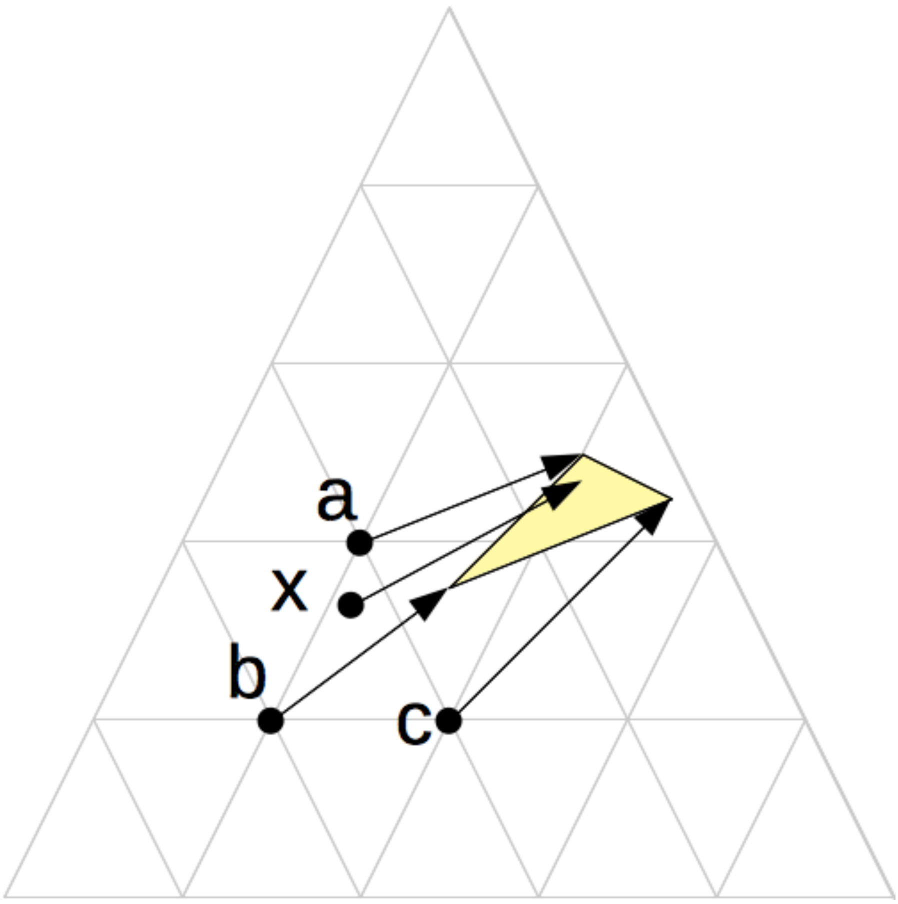
- Let be a function $\Phi: S \rightarrow 2^S$ such that
  - $\forall x$: $\Phi(x)$ is convex and non-empty
  - and *[closed graph condition]* if a sequence sequence $(x_0,y_0),(x_1,y_1),\ldots$ with $y_i \in \Phi(x_i)$ converges to (x,y) then $y \in \Phi(x)$
- **Kakutani's theorem:** Φ has a fix point



Idea of the proof: interpolation

- Let T be a triangulation of S
- Define a function Ψ:S→S
  - mapping every vertex x∈T to some arbitrary point in Φ(x)
  - and map every non-vertex point x∈S to the interpolation of Ψ(a), Ψ(b), Ψ(c) where abc is triangle of T containing x
- Now Ψ is continuous by interpolation, and we can apply Brouwer's fixed point theorem

□



Idea of the proof of Nash's Theorem

- Let X be a mixed stragegy profile
- Let f_i be the function computing the best responses of player i to X_{-i}
- Let g be the function mapping X to $f_1(X_{-1})\times\ldots\times f_n(X_{-n})$.
- Since the the image of f_i is non-empty and convex so is the image of g.
- Also g satisfies the closed graph property (believe me)
- So by Kakutani's fixed point Theorem there is a mixed strategy profile X with $X\in g(X)$, i.e. X is a mixed Nash equilibrium

2 player zero sum games

- Players have exactly opposite goals: $u_1=-u_2$
- Now goal is to hurt opponent
- x^* is a maxminimizer for player 1 if it is the best choice under the assumption that player 2 wants to hurt him as much as possible
  $\min_y U_1(x^*,y) \geq \min_y U_1(x,y)\ \forall x$
- y^* is maxminimizer for player 2 if
  $\min_y U_2(x^*,y) \geq \min_y U_2(x,y)\ \forall y$
- **Lemma**: $\max_x \min_y U_2(x,y)=-\min_y \max_x U_1(x,y)$

Photo (c): http://striepling.de

- **Proof**:
  $\max_x \min_y U_2(x,y) =$
  $\max_x \min_y -U_1(x,y) =$
  $\max_x -\max_y U_1(x,y) =$
  $-\min_x \max_y U_1(x,y)$
- **von Neumann**: (x^*,y^*) is a mixed Nash equilibrium iff x^*,y^* are maxminimizers
- maxminimizers can be found by linear programming

Summary

- stragegic games
- strategy profile
- best response
- pure Nash equilibrium
- mixed Nash equilibrium
- … do always exist for finite games
- 2-player zero sum games: finding mixed Nash equilibria is solving a linear program

Game Theory and Applications

2– Local search

November 2012, Departamento de Ingeniería Industrial, Univ. of Santiago, Chile

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What is the complexity of finding equilibria ?

- finding a pure Nash equilibrium is polynomial when the payoffs are given as tables
- But when the payoffs are fixed functions, finding pure Nash equilibria is often an FNP-hard problem
- finding a mixed Nash equilibrium is PPAD-hard
- finding a mixed Nash equilibria is polynomial for 2 player zero sum games

- We just need to check a linear number of strategy profiles
- FNP=class of functions f, such that checking f(x)=?y is in NP
- PPAD is believed to be disjoint from P and from NP-hard
- uses von Neumann’s minimax Theorem

Some complexity classes

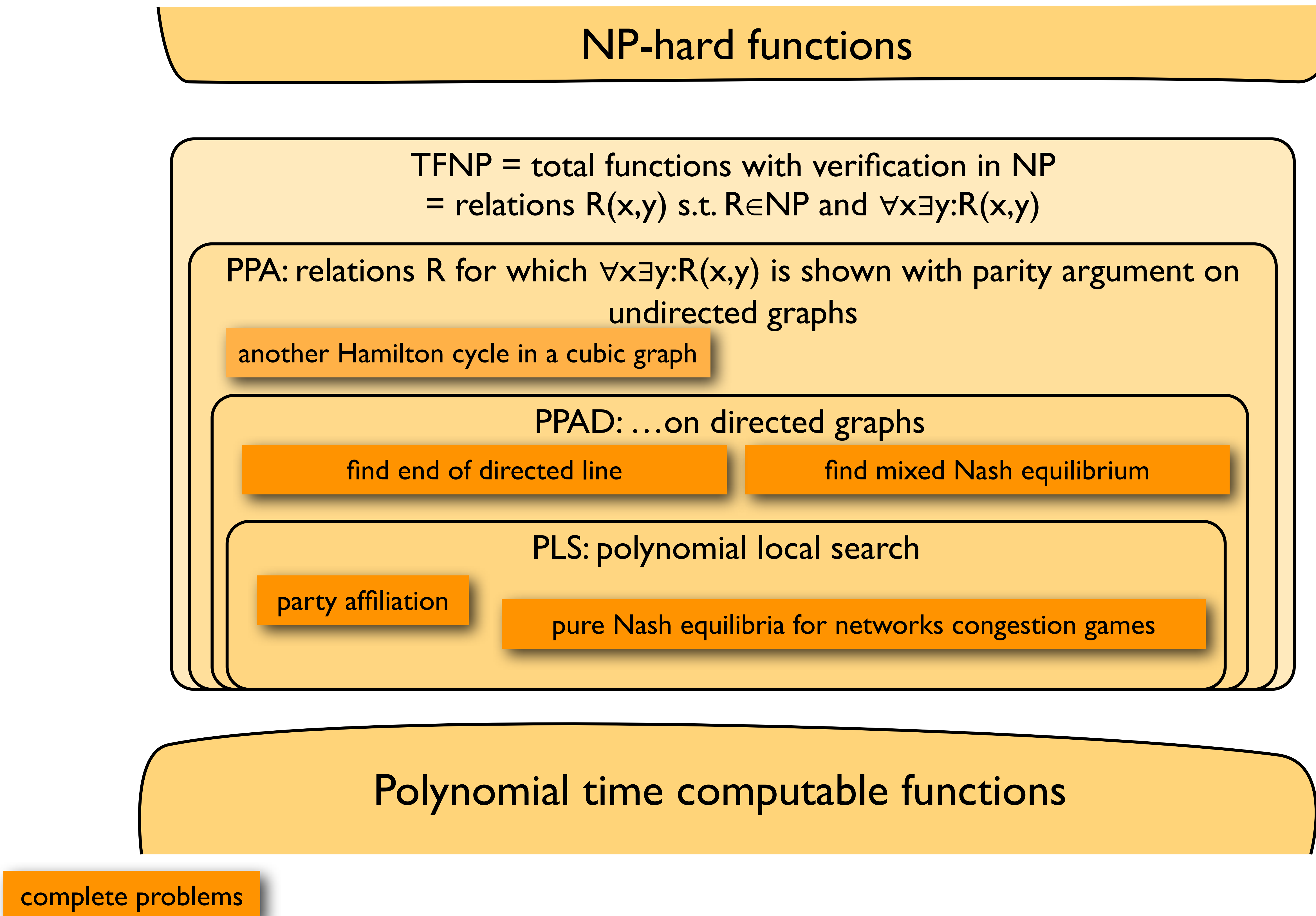
NP-hard functions

TFNP = total functions with verification in NP
= relations R(x,y) s.t. R∈NP and ∀x∃y:R(x,y)

for example
x=strategic game,
y=mixed Nash equilibria

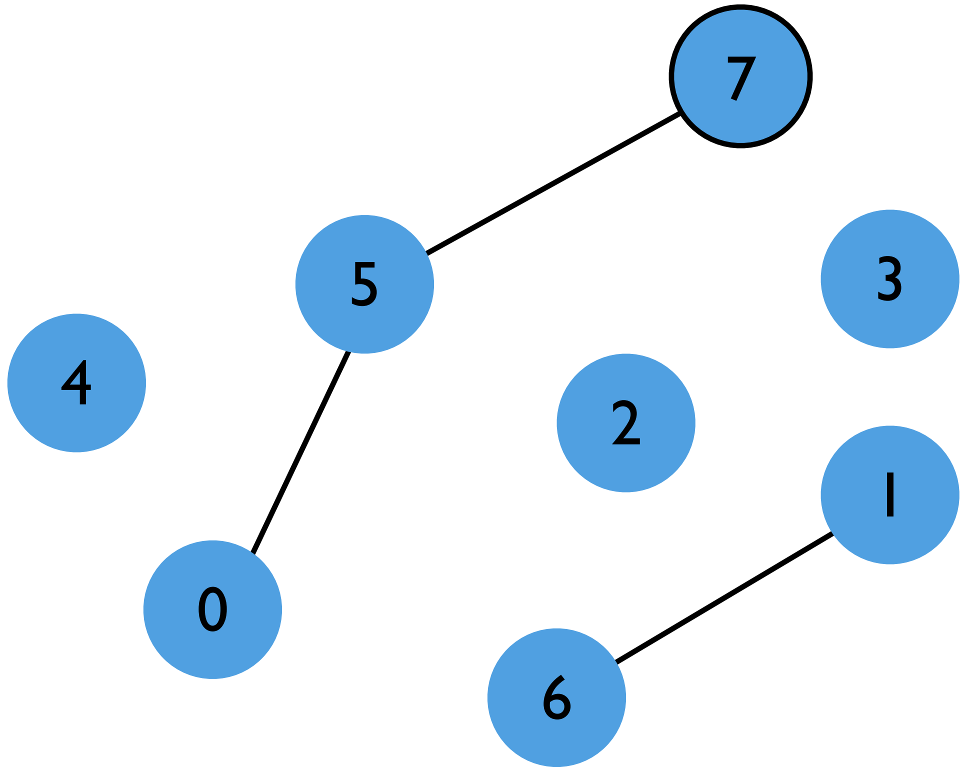
Polynomial time computable functions

Some complexity classes



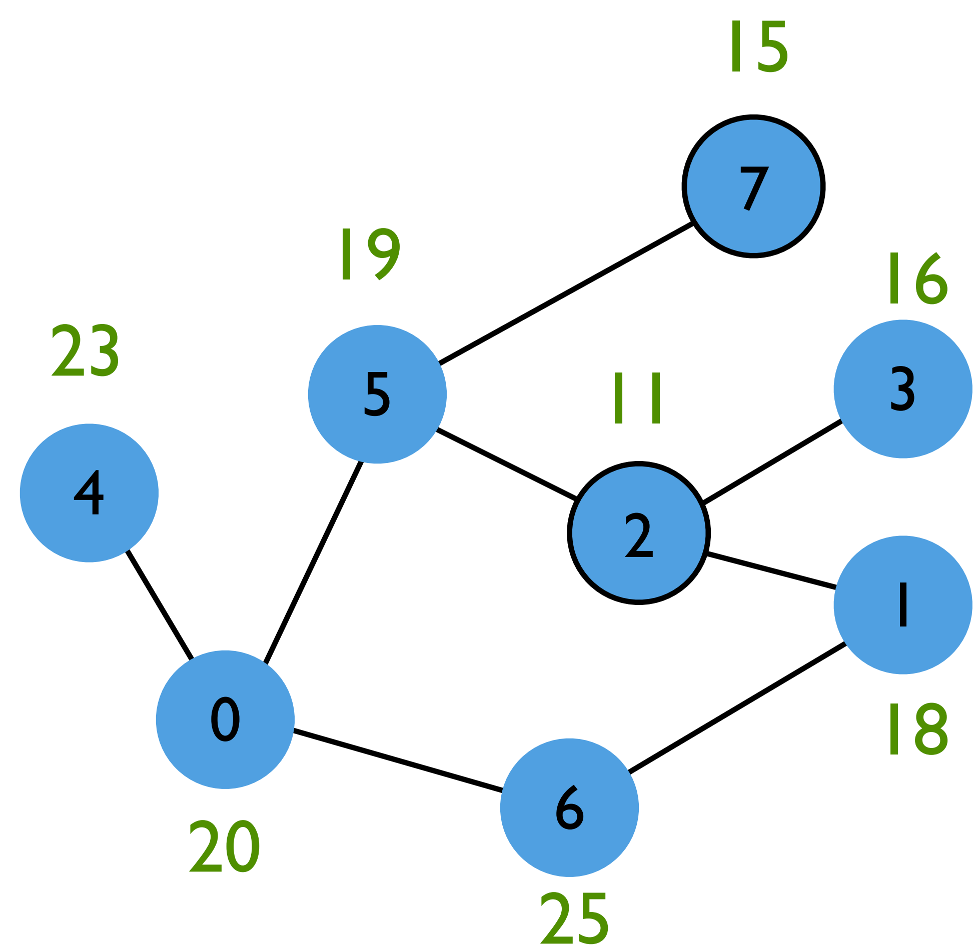
Polynomial Parity Argument
PPA

- Class of functions f:x→y (*or relations R(x,y)*)
- s.t. x defines a graph of degree 2 described implicitly by a polynomial time function
  g : (x,u)→neighbors (up to 2)
  g(x,0)=(5)
  g(x,5)=(0,7),....
- with |g(x,0)|=1, i.e. 0 has degree 1
- f(x) is the end of the path starting at 0



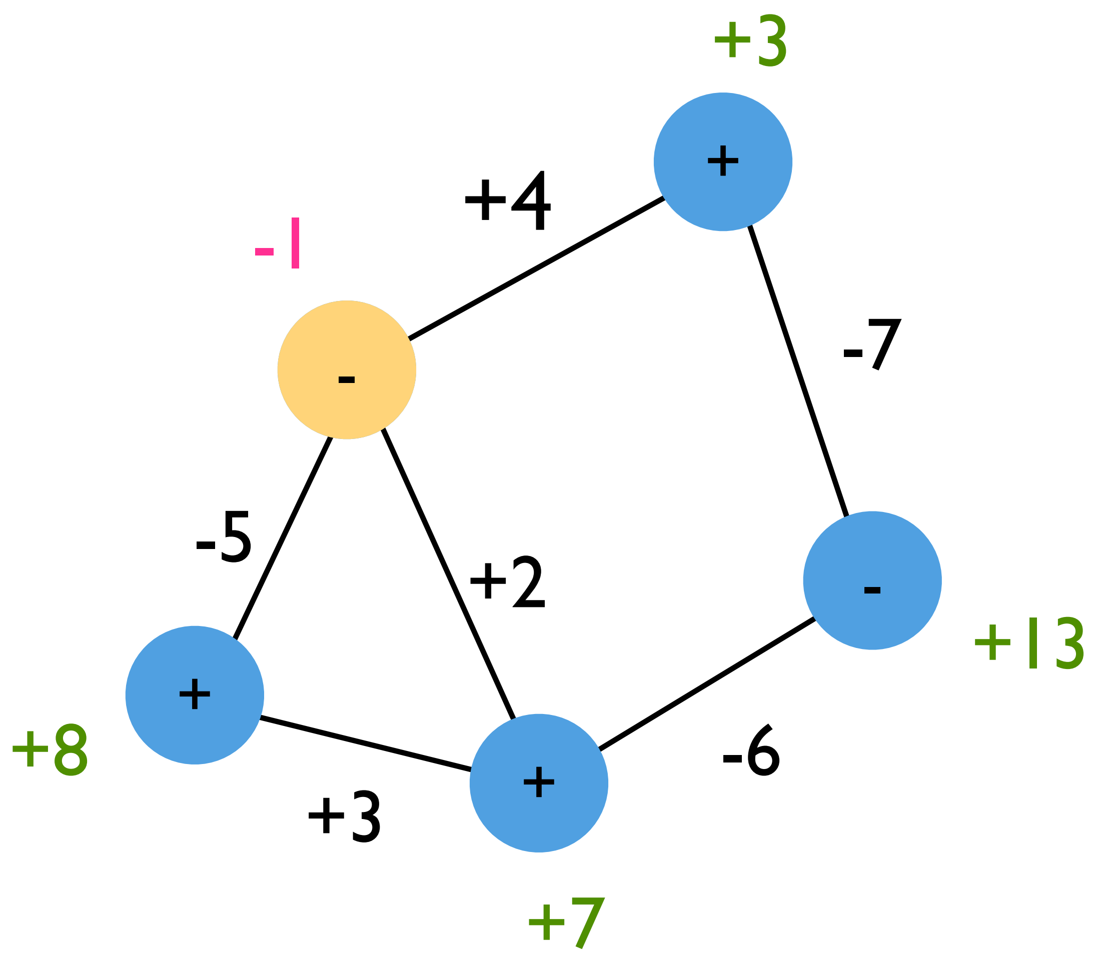
Polynomial Local Search
PLS

- Class of functions f:x→y (*or relations R(x,y)*)
- s.t. x defines a weighted graph of polynomial degree
  described implicitly by 2 polynomial time functions
  g :(x,u)→neighbors
  w:(x,u)→weight of u
- f(x) is a vertex of weight ≤ the weights of its neighbors (local minima)



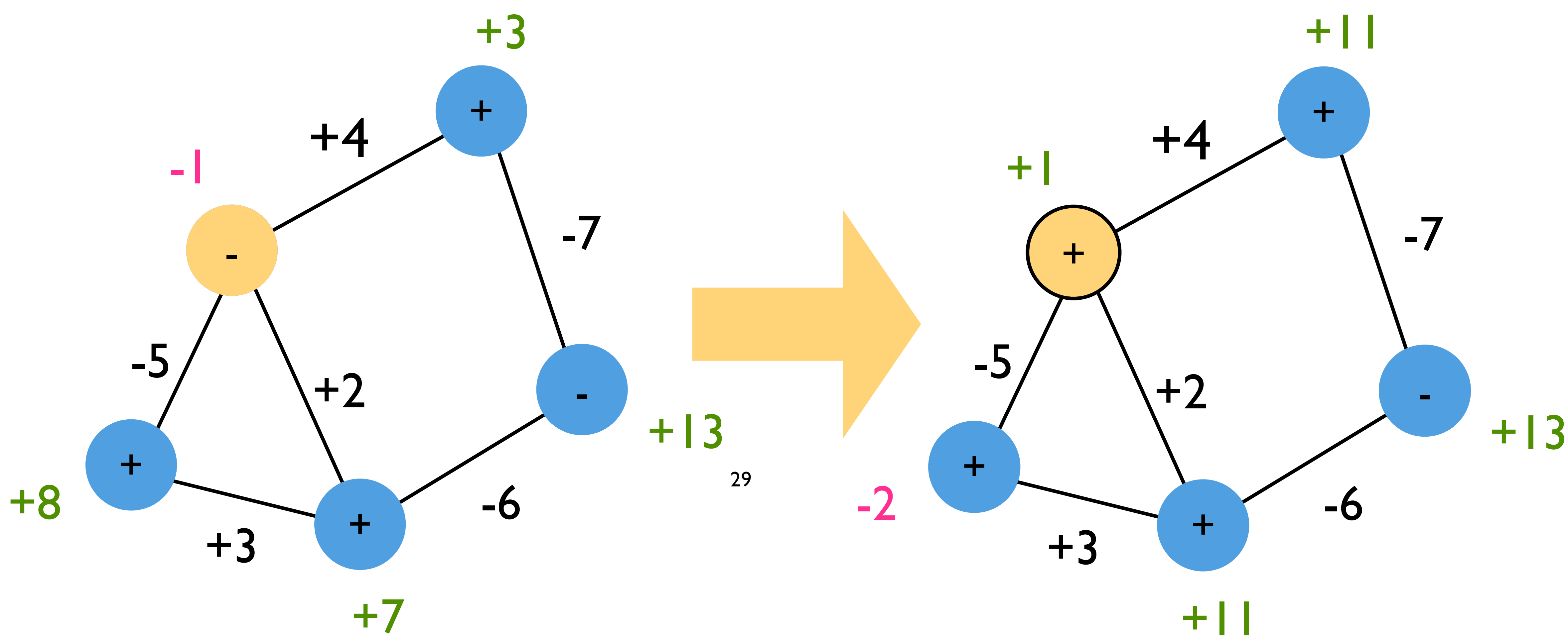
Party Affiliation Game ∈PLS

- a symmetric relation matrix $M \in \mathbb{Z}^{k \times k}$ with $M_{ii}=0$ ($\forall i$)
- for a strategy profile $s \in \{-1,+1\}^k$ the payoff of player i is
- $u_i := \sum_j s_i M_{ij} s_j$
- s is a Nash equilibrium iff $u_i \geq 0$ ($\forall i$)



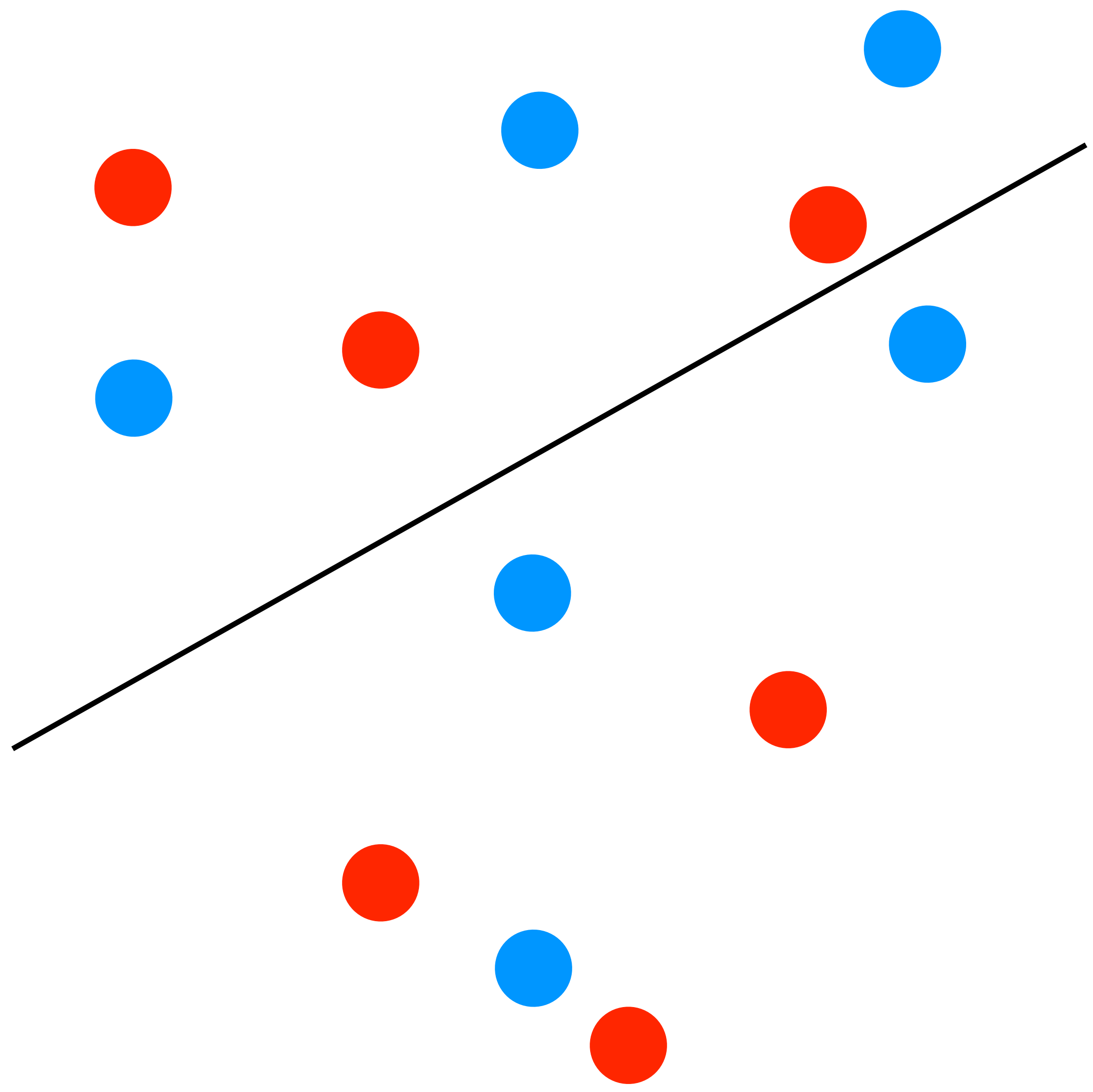
Best response dynamics

- **local search:** while there are unhappy players, change the strategy of an arbitrary unhappy player
- the potential function $\sum u_i$ increases strictly and is upper bounded by $\sum |M_{ij}|$
- therefore the best response dynamics never cycles



Ham Sandwich Cut ∈PPAD

- Here: a simple geometric 2-dimensional version
- Given n red points and n blue points in the plane
- Find a hyperplane that separates the red points in half and also the blue points in half
- This simple geometric 2-dimensional version can be solved in linear time [Lo,Steiger'1990]
- For d dimensions, best algorithm runs in $O(n^{d-1})$
- **Open:** is this problem PPAD-complete?



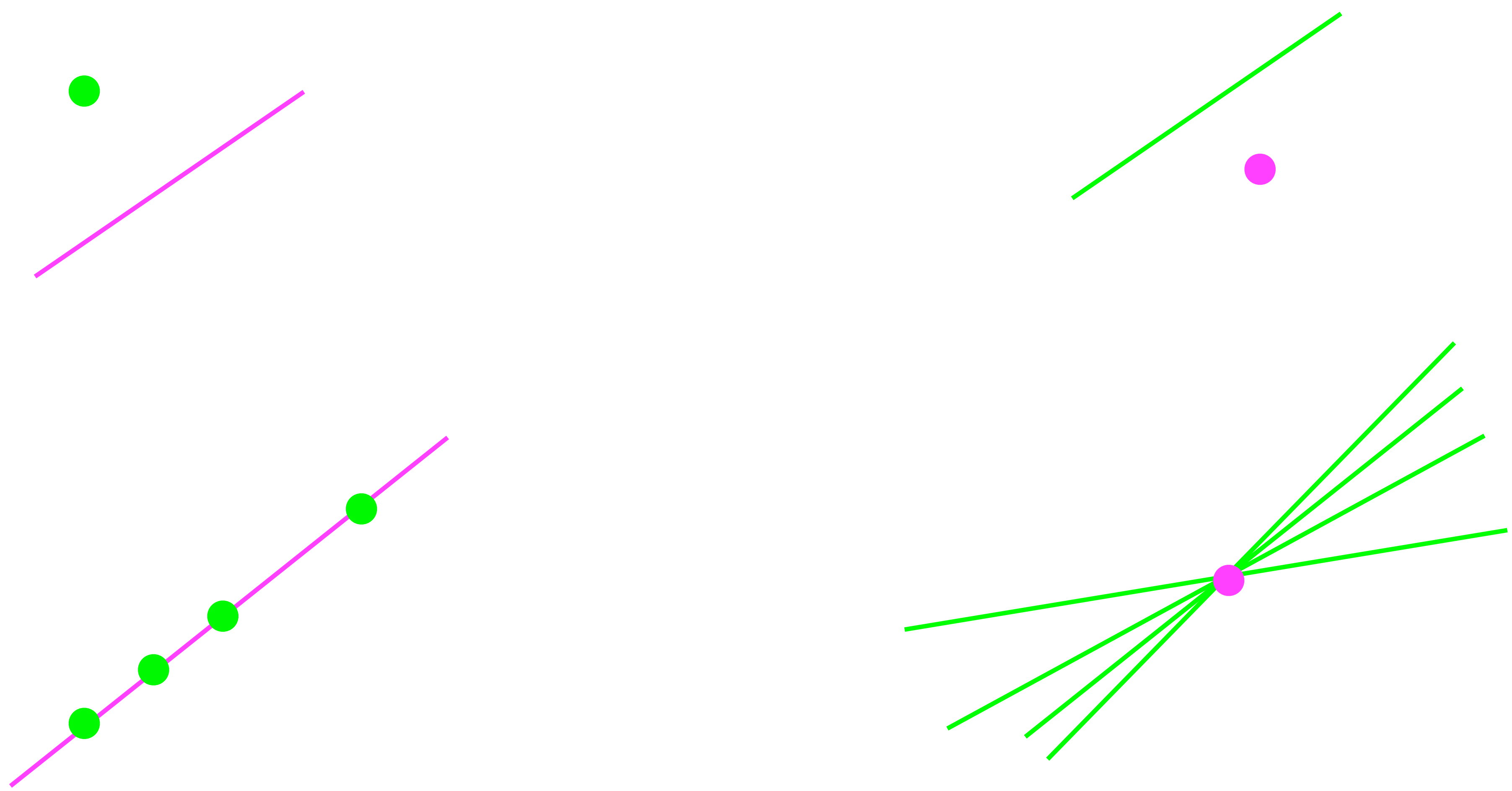
Idea of proof: geometric duality

Primal

- point (a,b)
- point p is above line l iff…

Dual

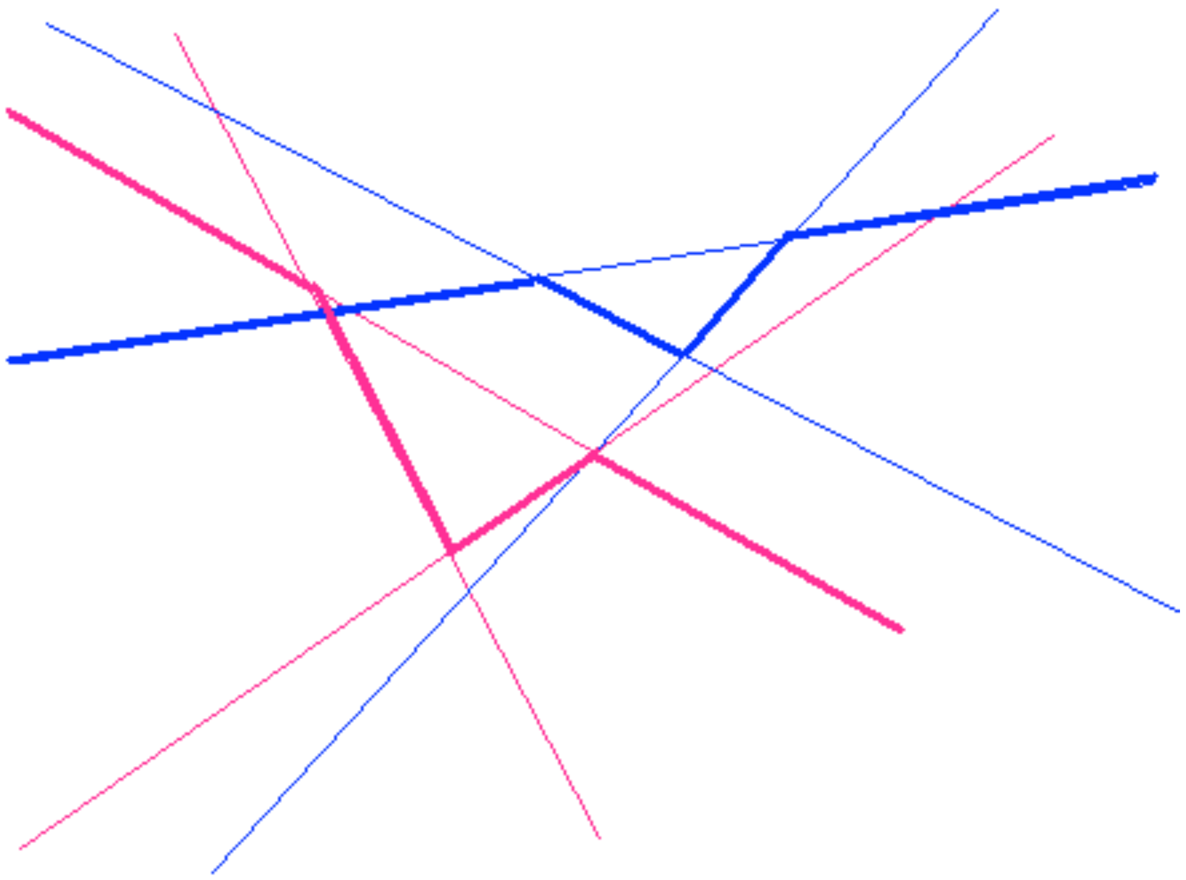
- line y=ax+b
- … iff line p* is above point l*



Median of lines

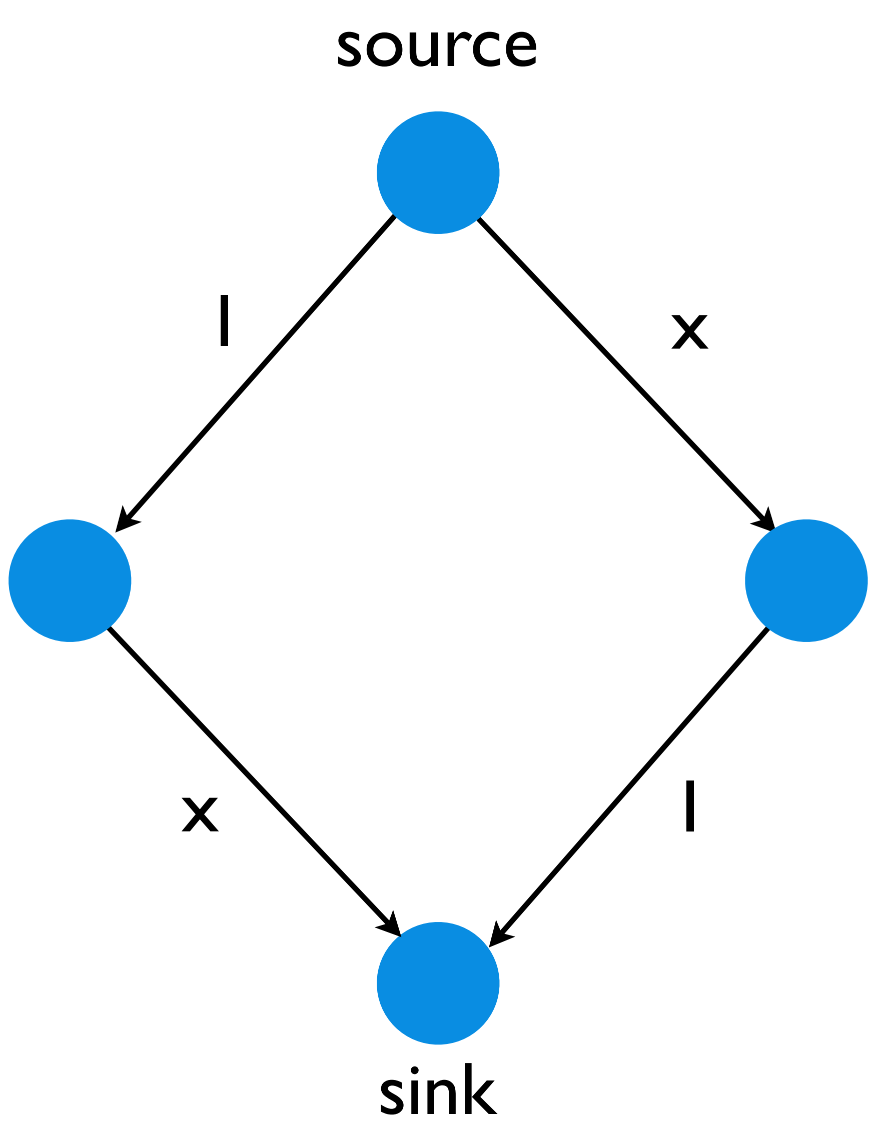
- We have an arrangement of n red lines (say n is odd)
- And consider the median level of these lines (set of points such that half of red lines are above and half below)
- These points belong to one of the lines (assumption n is odd)
- Interesting property: At x=-∞ and x=+∞ the median belongs to the same line
- Same for the blue lines
- (!) The medians intersect an odd number of times

Picture (c) : http://cgm.cs.mcgill.ca/~athens/cs507/Projects/2002/DanielleMacNevin/



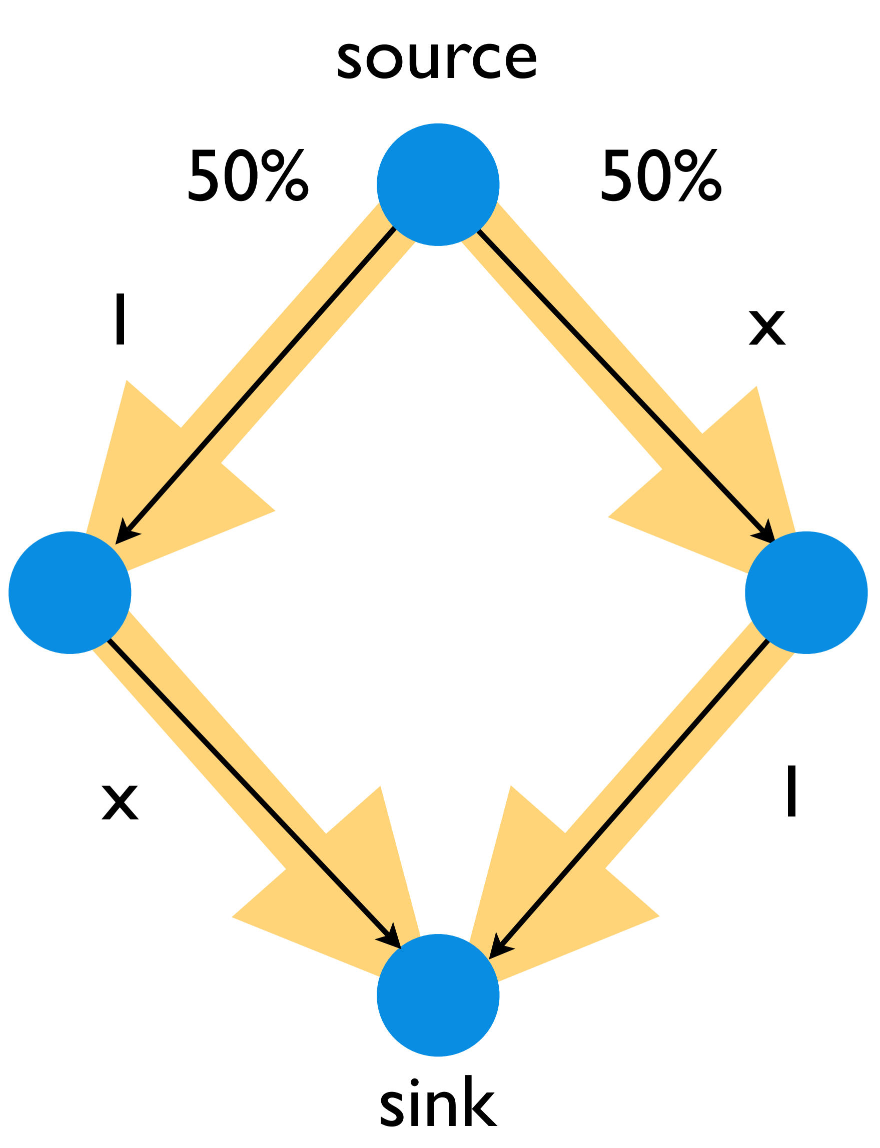
Congestion games: the Braess paradox

- [Braess, 1968]
- time spent on road = cost of driver
  - 0=broad highway
  - 1=road with constant slow speed
  - x=road with traversal time linear in the number of users



Optimum

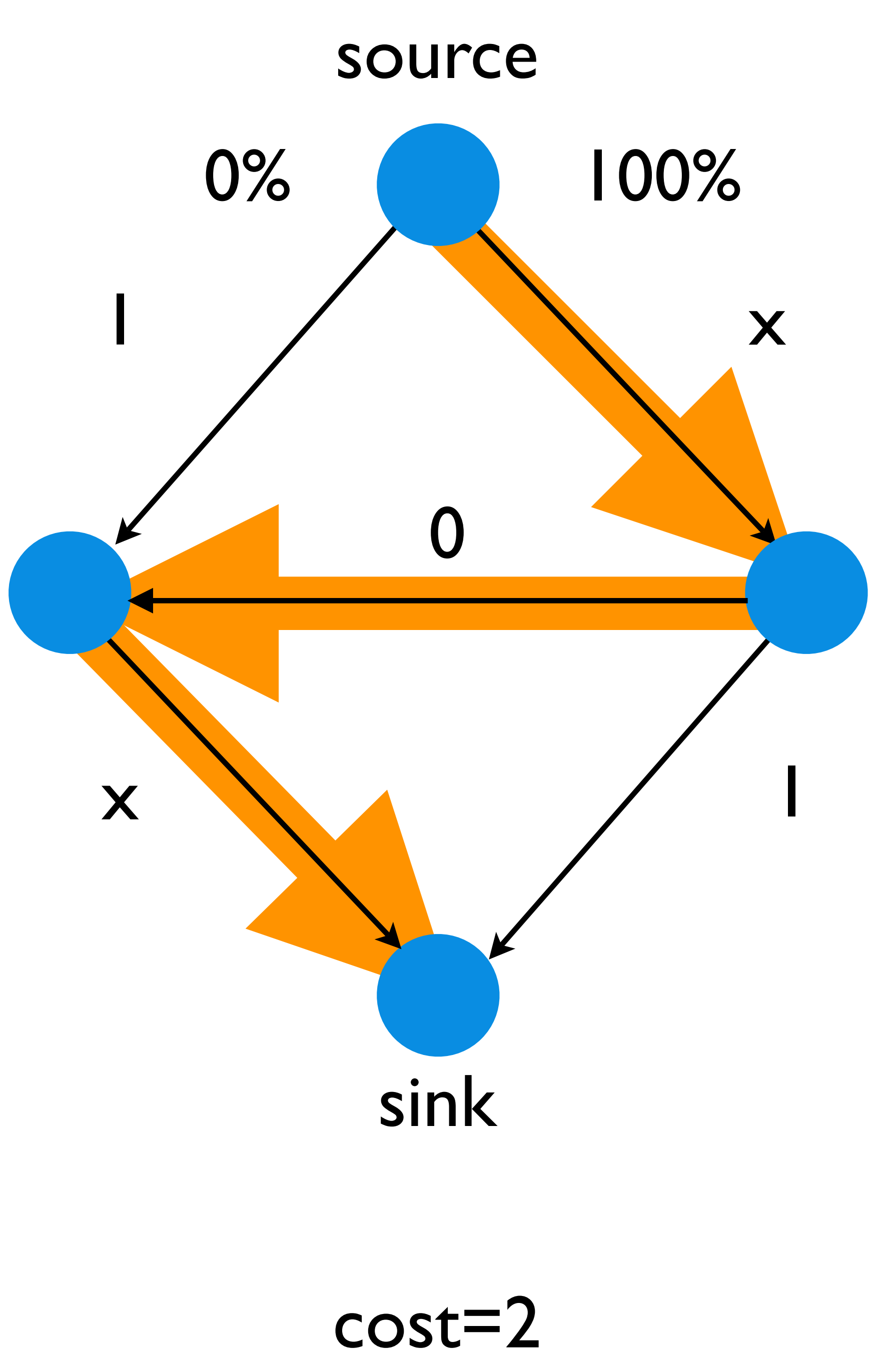
- 100 drivers
- Equal split of traffic is the unique Nash equilibria with cost 1.5
- which is also the optimum for social cost = total drivers cost



cost=1.5

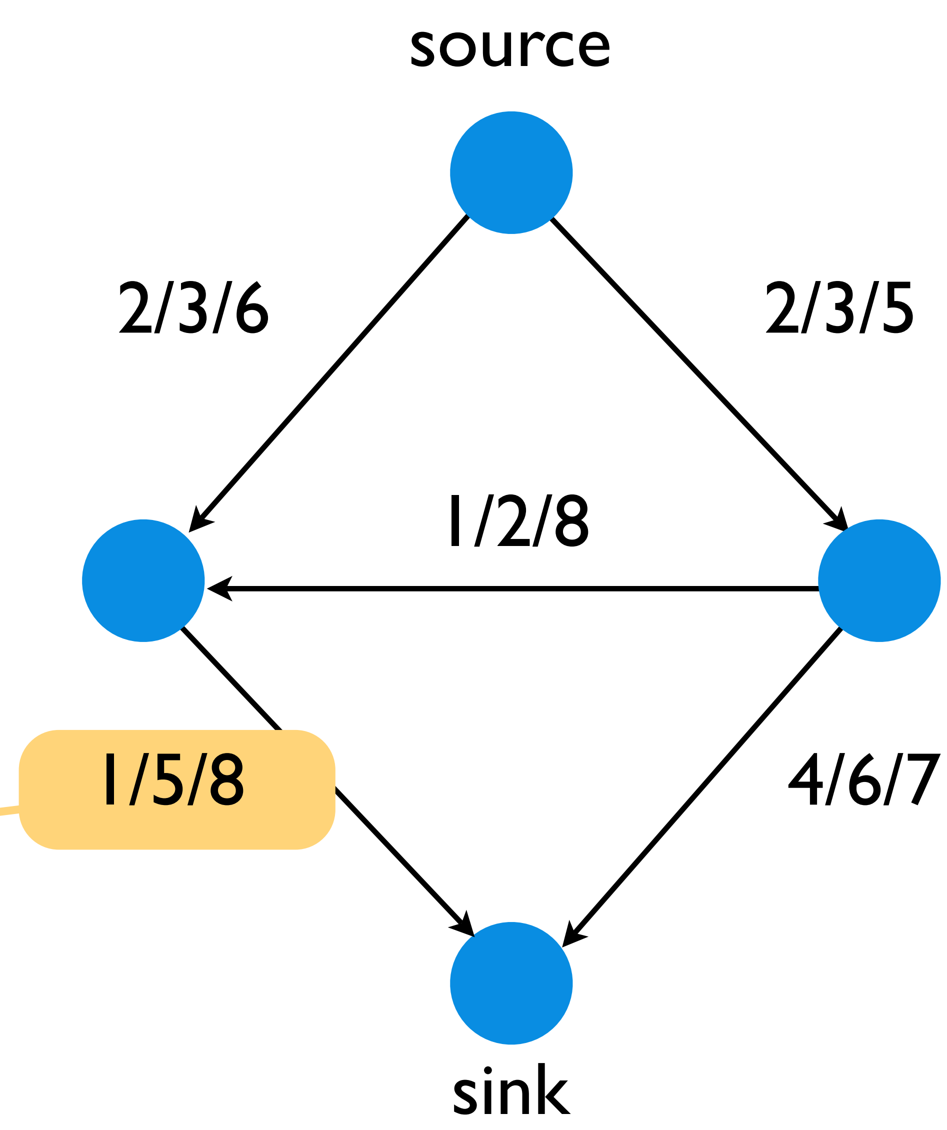
Build a highway

- social cost =
  total drivers cost
- Unique Nash equilibrium has cost 2
- **which is worse !**
- this phenomen is not rare



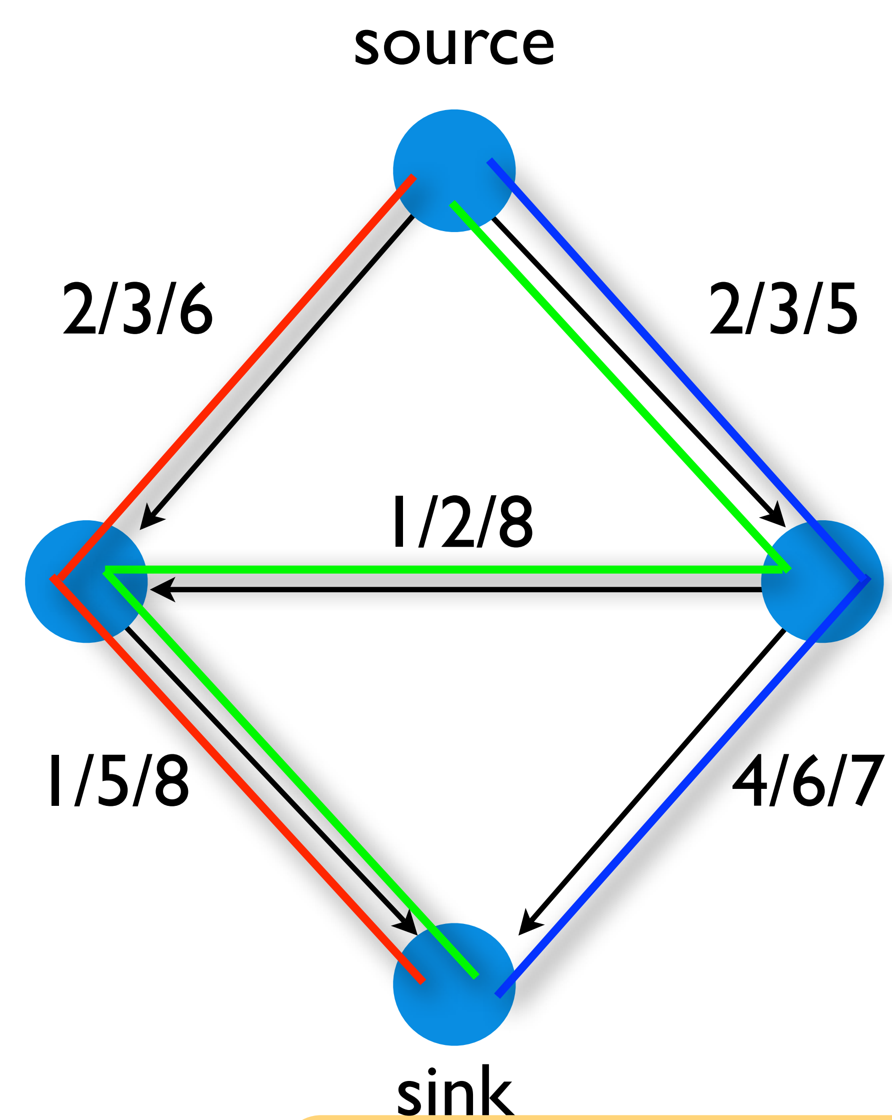
Network congestion game

- Player want to route from a source to a destination
- *Symmetric game* = same source, same destination for all players
- Link congestion depends on the number of users routing through it : congest. for 1,2,3 users
- The cost of a player is the sum of the congestion over all used links



Formal definition

- Congestion game = $(N,R,(S_i)_{i\in N},(d_r)_{r\in R})$
- $N=\{1,...,n\}$ set of players
- $R=\{1,...,m\}$ set of resources
- $S_i \subseteq R$ set of strategies of player i
- $d_r: \mathbb{N} \to \mathbb{Z}$ cost function for resource r
- given strategy profile $s=(s_1,..,s_n)\in S_1\times ...\times S_n$ nb of players using r is $n_r := |\{i : r\in s_i\}|$
- cost of player i is $\sum_{r\in si} d_r(n_r(s))$



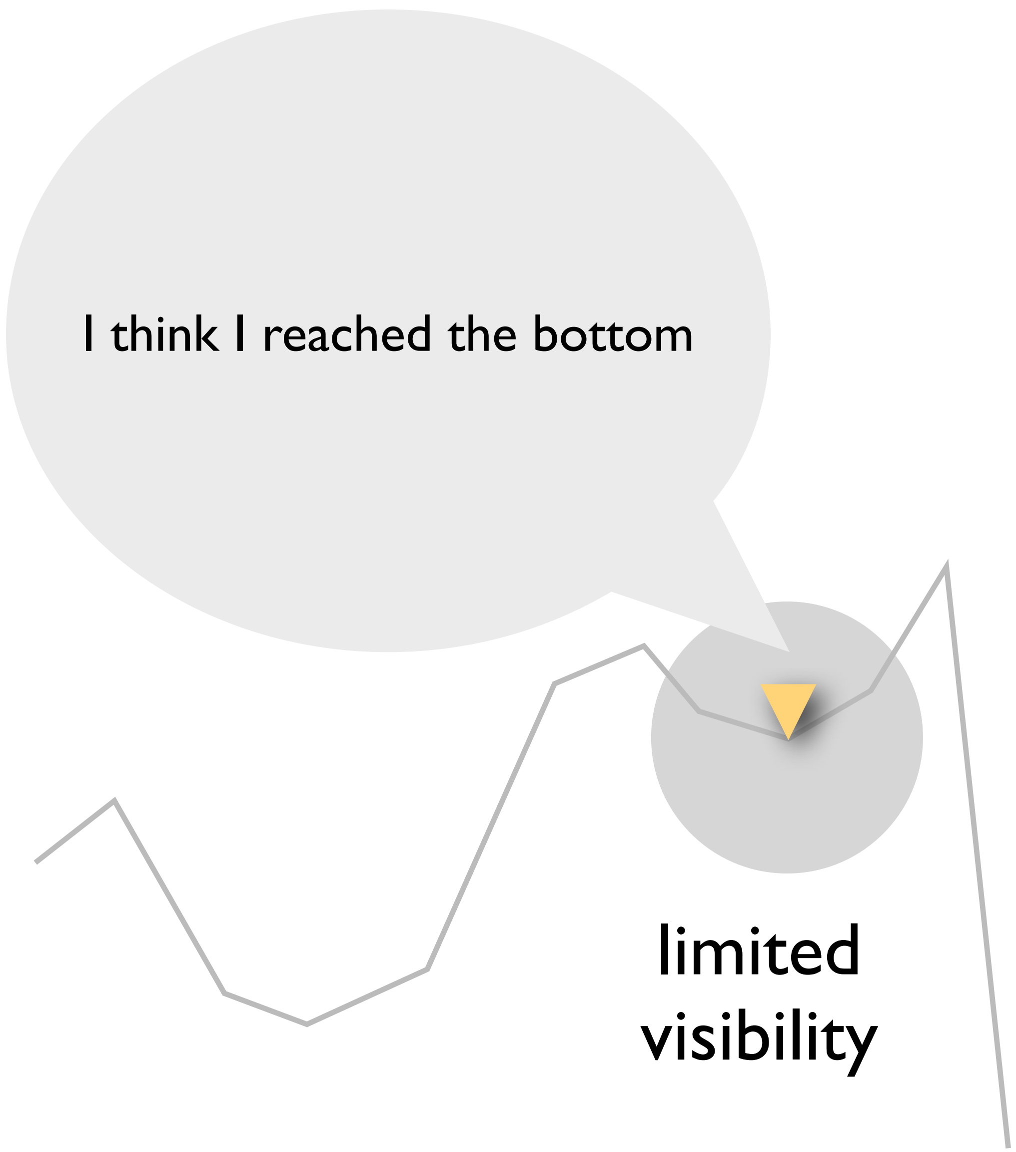
costs of

player 1=2+5 =7
player 2=3+1+5 =9
player 3=3+4 =7

Potential games

- A function $\Phi: S_1 \times \ldots \times S_n \rightarrow Z$ is an *exact potential function* if whenever a player decreases its cost by Δ, the function Φ decreases also by Δ.
- In that case the game is called a *potential game.*
- Every potential game can be turned in a congestion game. [Monderer,Shapley'96]

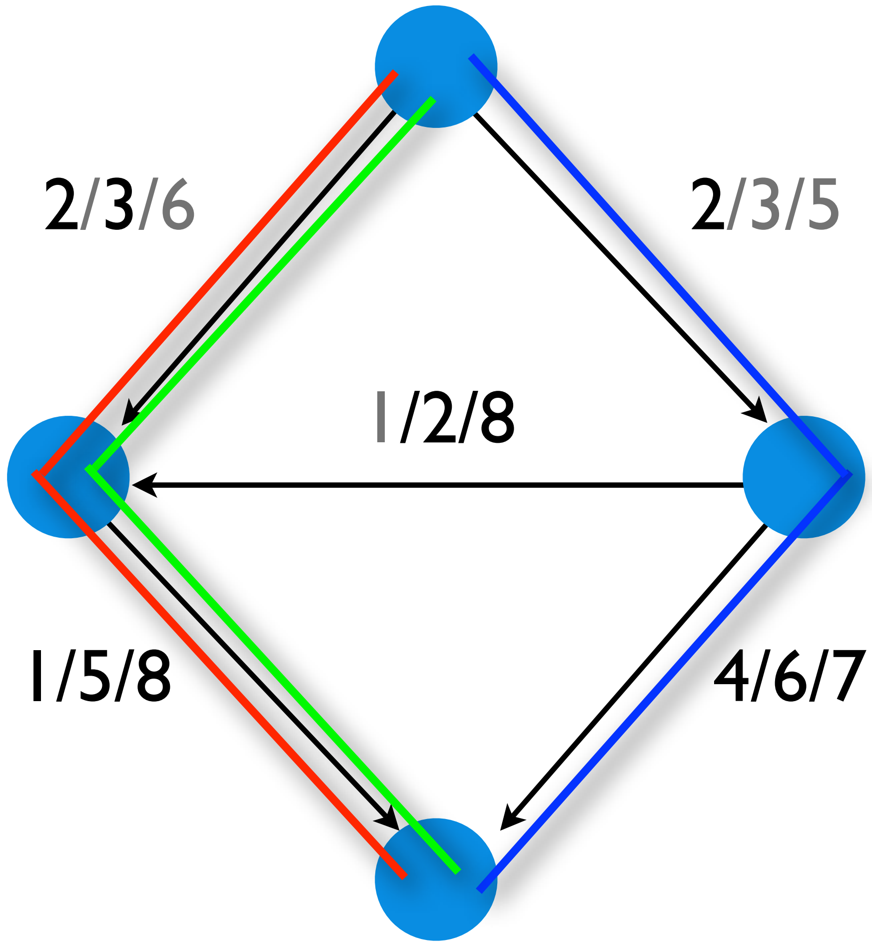
Connection to Local Search



Existence of pure Nash Eq.

- For every congestion game the best response dynamics converges in finite steps [Rosenthal'73]
- In fact he shows that it is a potential game with

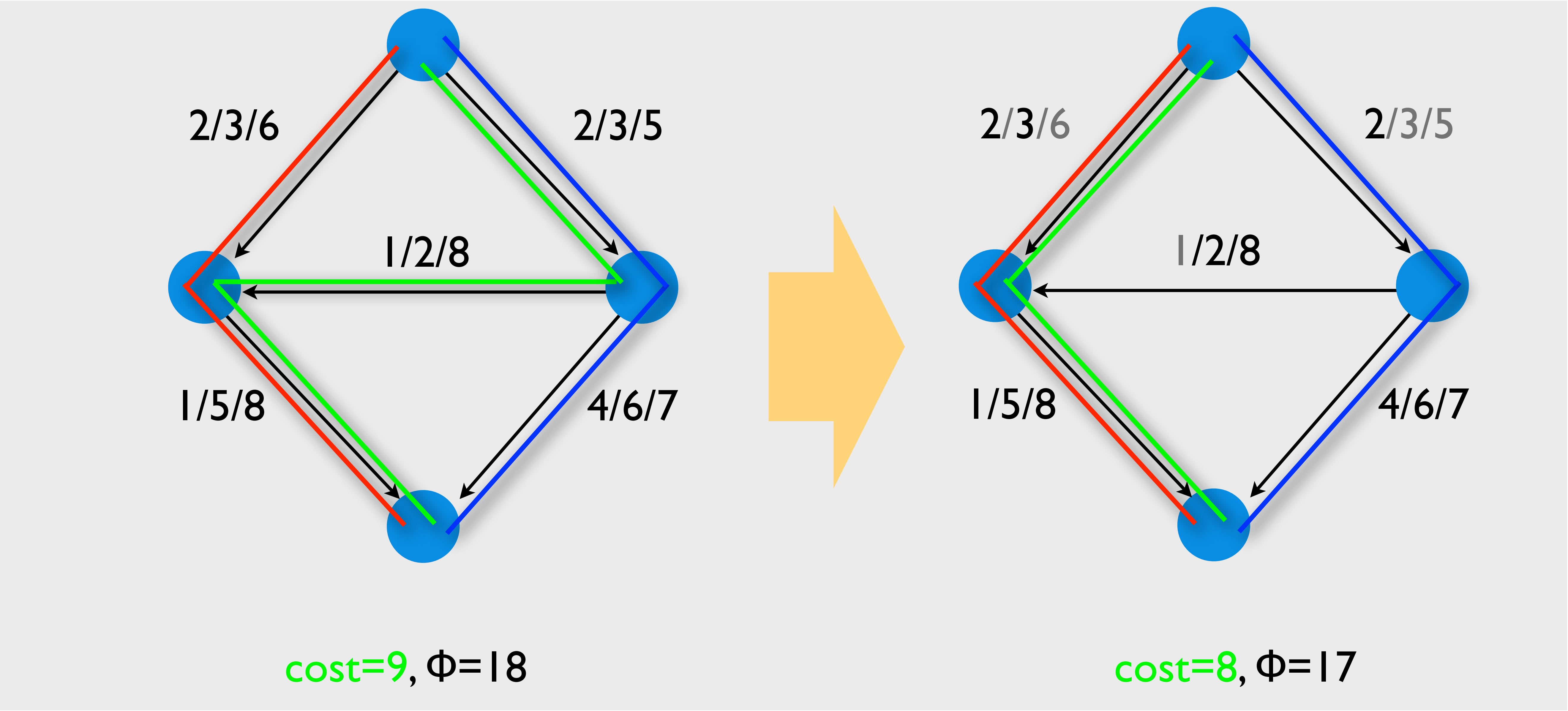
$$\Phi=\sum_{r\in R}\sum_{j\leq n_r} d_r(j)$$



$\Phi=2+3+1+5+2+4=17$

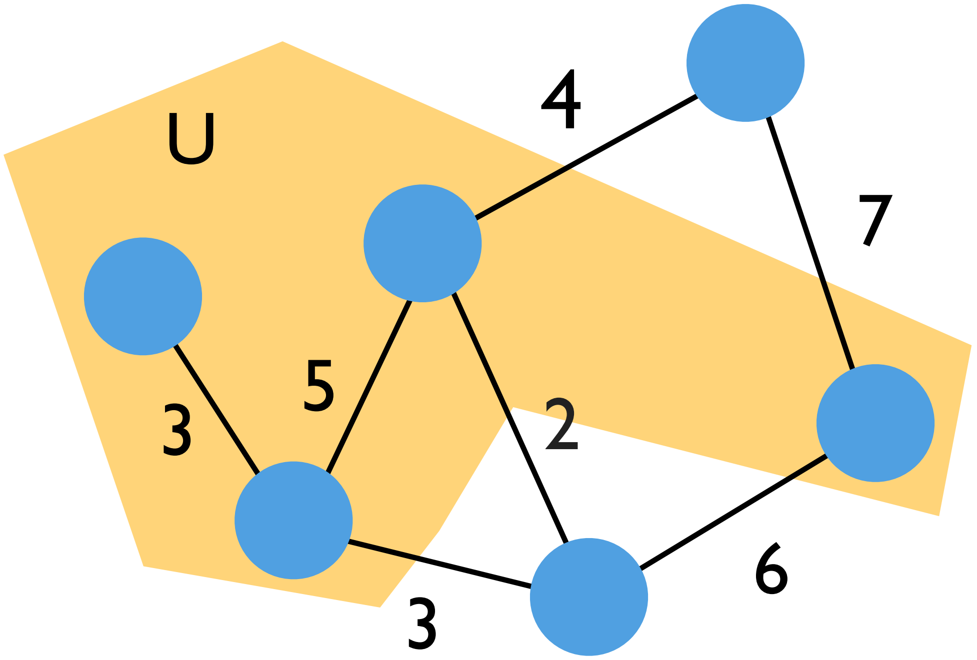
Φ is an exact potential fct

- therefore the game has always a pure Nash equilibrium. But how to find one?



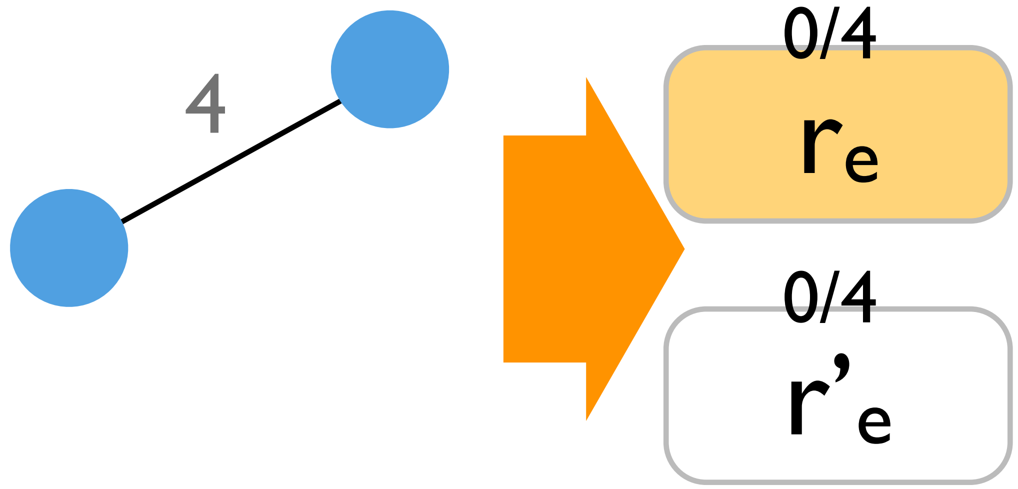
Computing Nash eq. for symmetric congestion games is PLS-complete

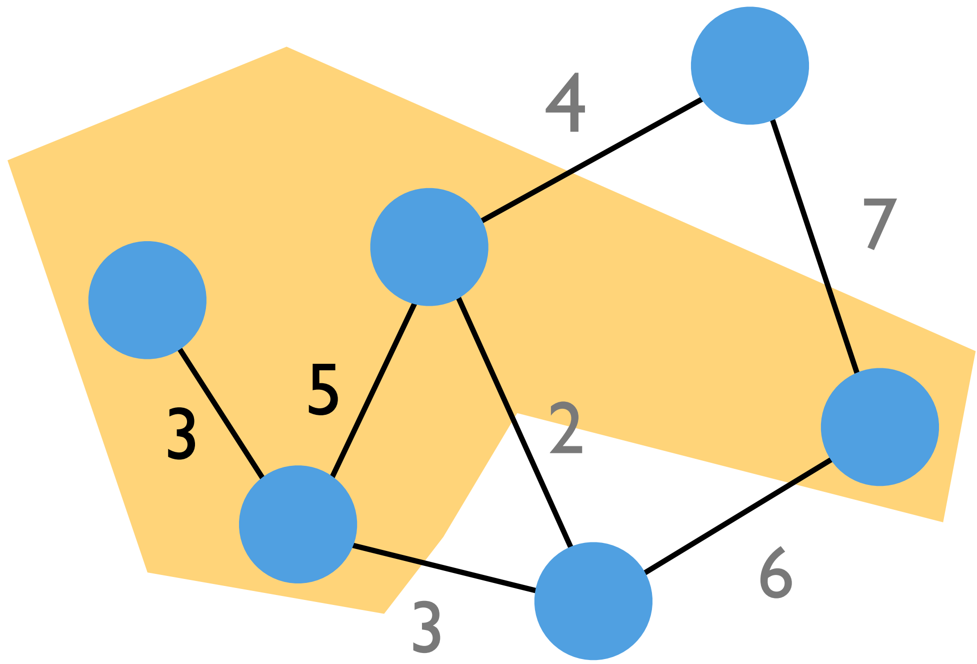
- [Fabrikant,Papadimitriou,Talwar'04] but this proof is from [Vöcking'06]
- Reduction from MaxCut (also PLS-complete): Given a graph G(V,E) $w:E\to\mathbb{R}$ find a set U such that $\sum_{u\in U, v\in (V\setminus U)} w(u,v)$ cannot be improved by a *1-flip*: add or remove a single vertex from U.



The reduction

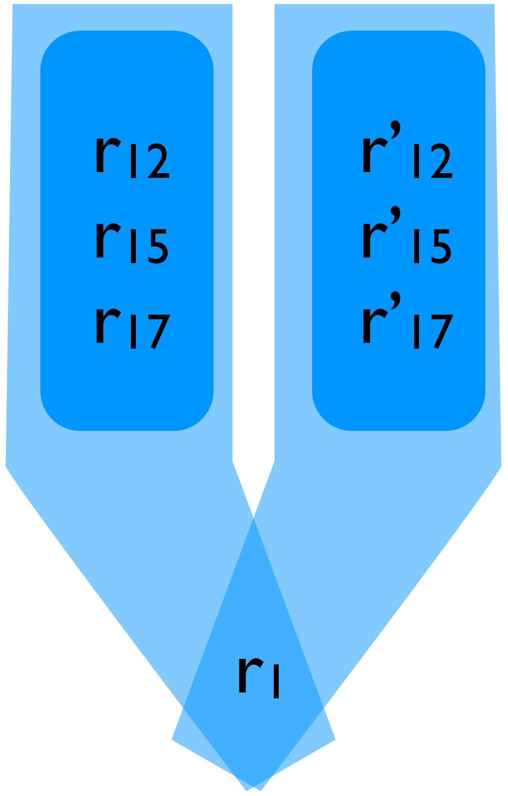
- ∀ edge e of weight w, there are two resources r_e and r'_e of cost 0/w/w/...
- players are nodes, two strategies for v $\{r_{(u,v)}: u\}$ or $\{r'_{(u,v)}: u\}$
- Nash eq. in the game= local maxima in MaxCut





Make the game symmetric

- add resources $r_1,\dots,r_n$ with cost $0/\infty/\infty/\dots$
- Set $S=\{s \cup \{r_v\} : v, s \in S_v\}$
- Now every player has the same strategy set S (symmetric game)
- In a Nash eq. exactly one player chooses one of
  $\{r_v\} \cup \{r_{(u,v)} : u\}$ or
  $\{r_v\} \cup \{r'_{(u,v)} : u\}$



Summary

- classes between P and NP-hard
- …contain party affiliation, ham sandwich cut, mixed Nash equilibria
- congestion games = potential games : do always have pure Nash eq.
- in general following the best response dynamics is the unique way to find a pure Nash equilibrium

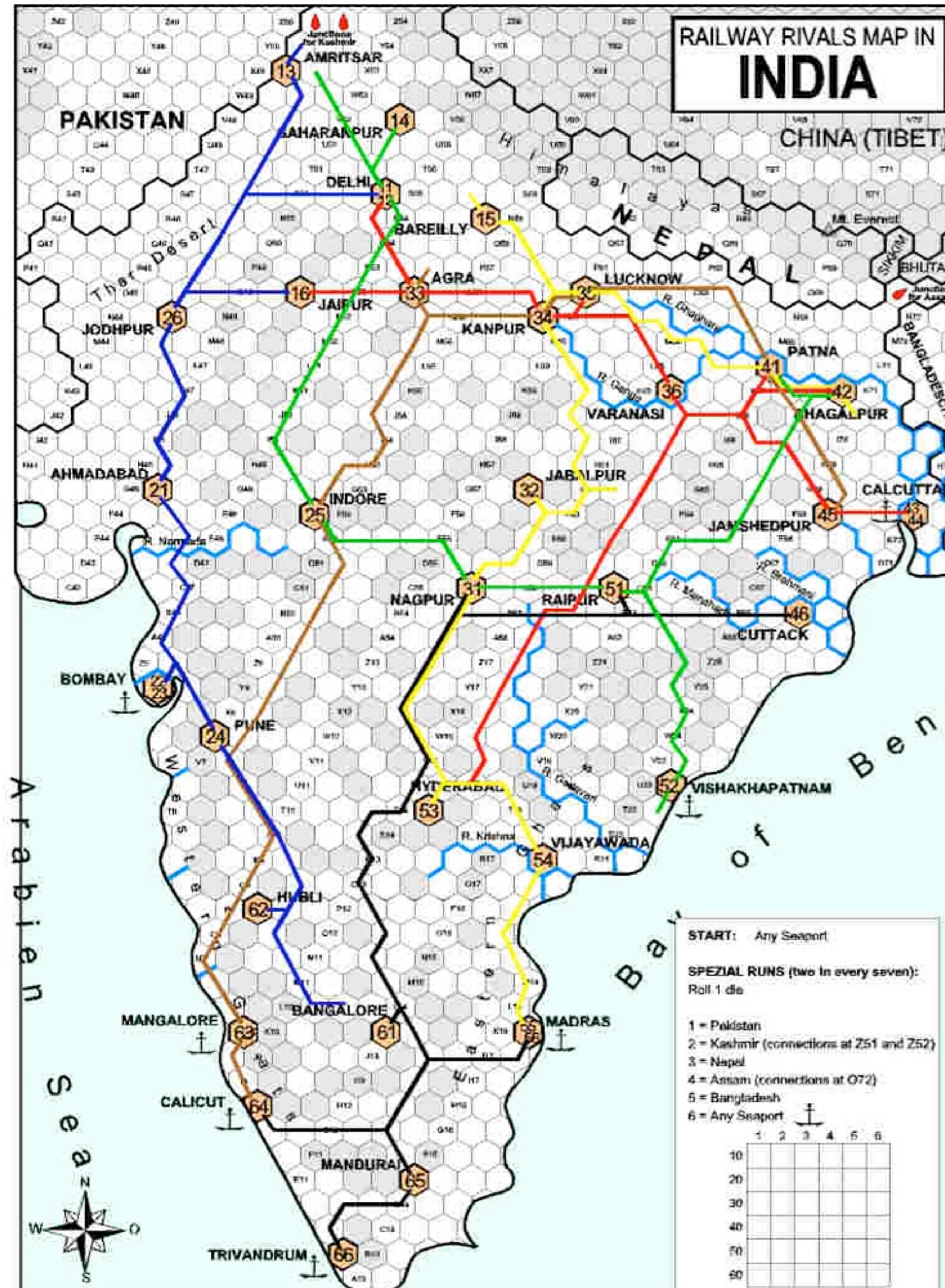
Game Theory and Applications

3– Network creation games

November 2012, Departamento de Ingeniería Industrial, Univ. of Santiago, Chile
Christoph Dürr

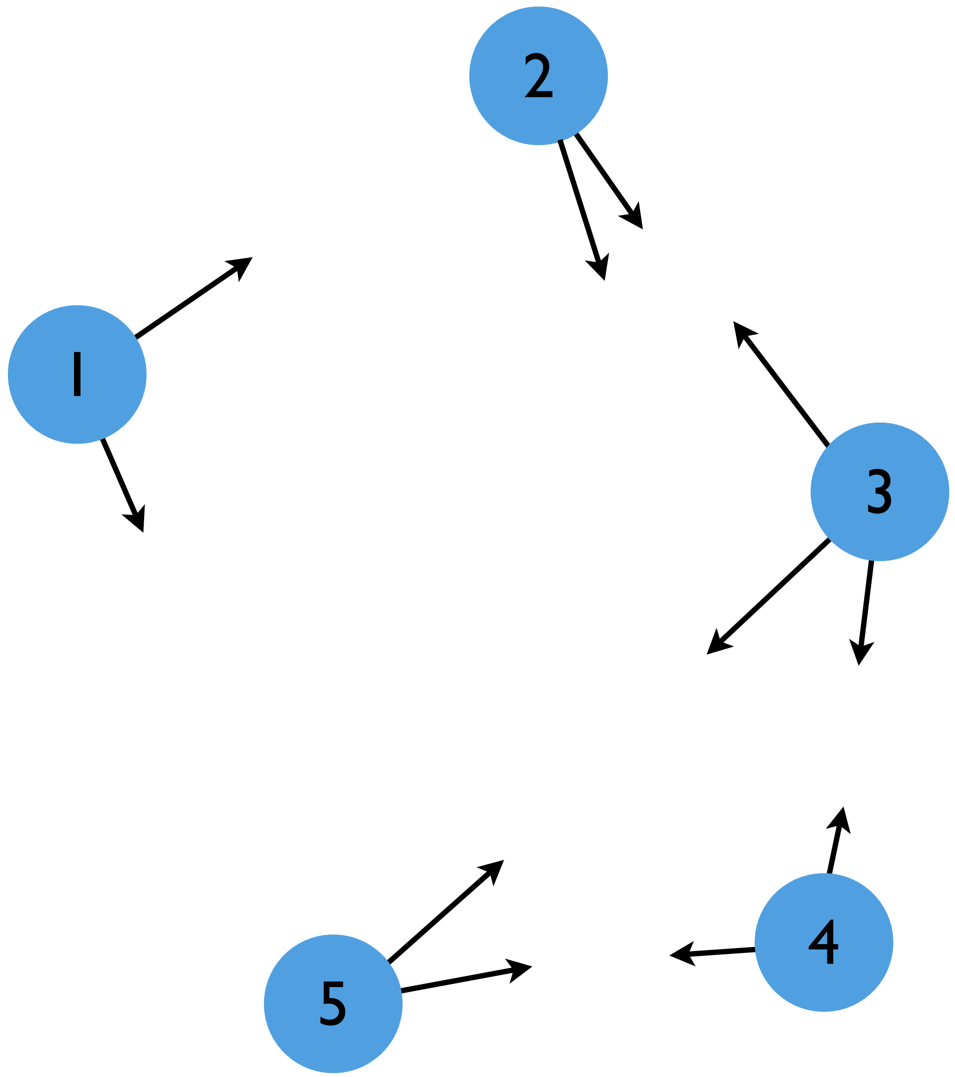
Dampfross

- plays on a map
- 1st round, players pay for building tracks (strategy)
- 2nd round, players pay for using the network



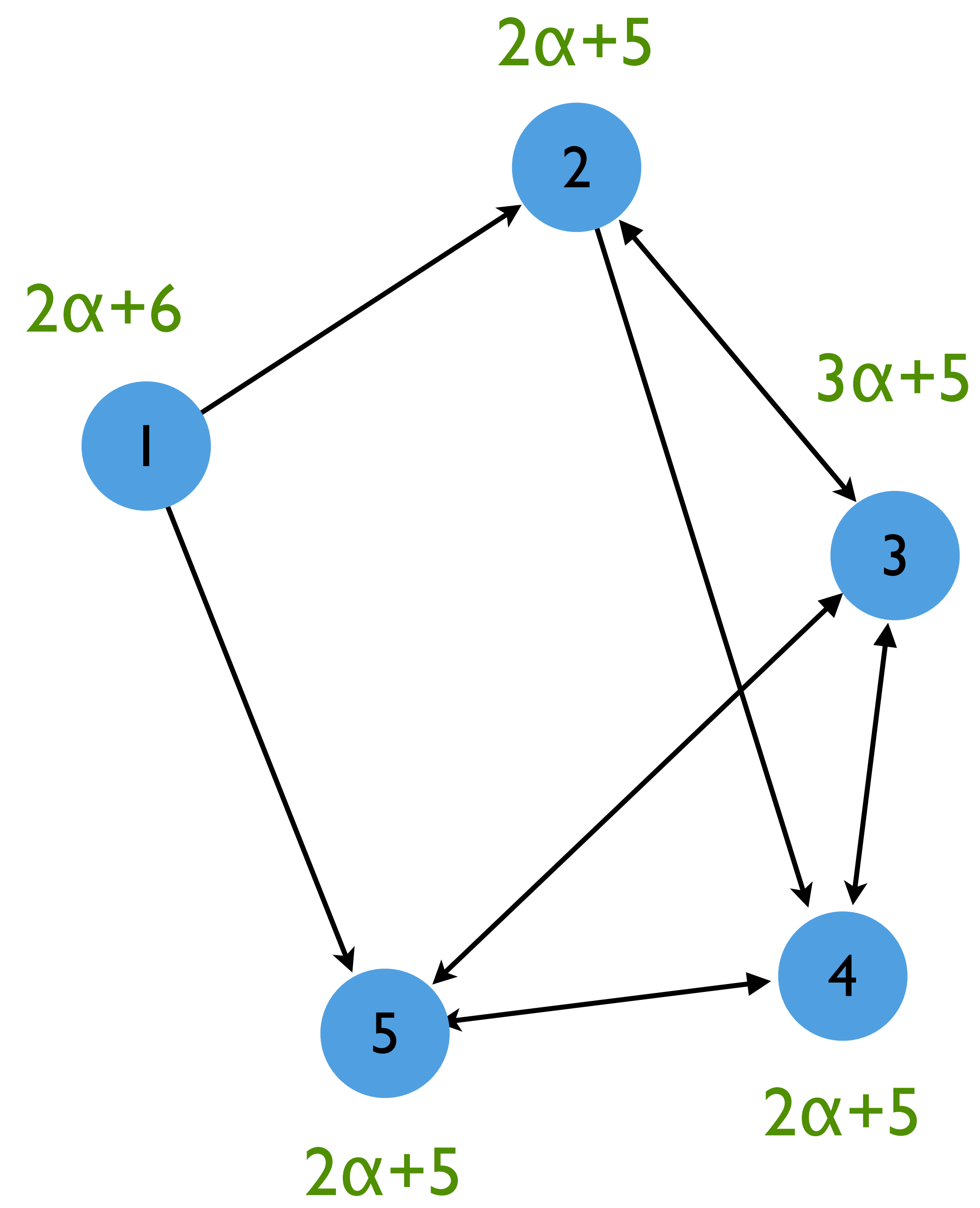
Network creation game

- players are nodes
- a strategy is a set of other players to connect to
- **stage 1**: connections are build and billed α to the origins and can be used in both directions
- $\alpha \geq 0$ is the parameter of the game



Network creation game

- edges can be used in both directions, orientation shows only who pays for it
- **stage 2**: every user pays in addition the total shortest path length to all other users
- is this a good network?



Formal definition

- Strategy for player u (vertex u) is set of vertices v, it generates arcs (u,v)
- A directed graph is a strategy profile
- The cost of player u (vertex u) is

  $\alpha \cdot \deg(u) + \Sigma_v\, d(u,v)$

  where deg is the outdegree and d is the distance in the undirected graph
- social cost = sum of individual costs =

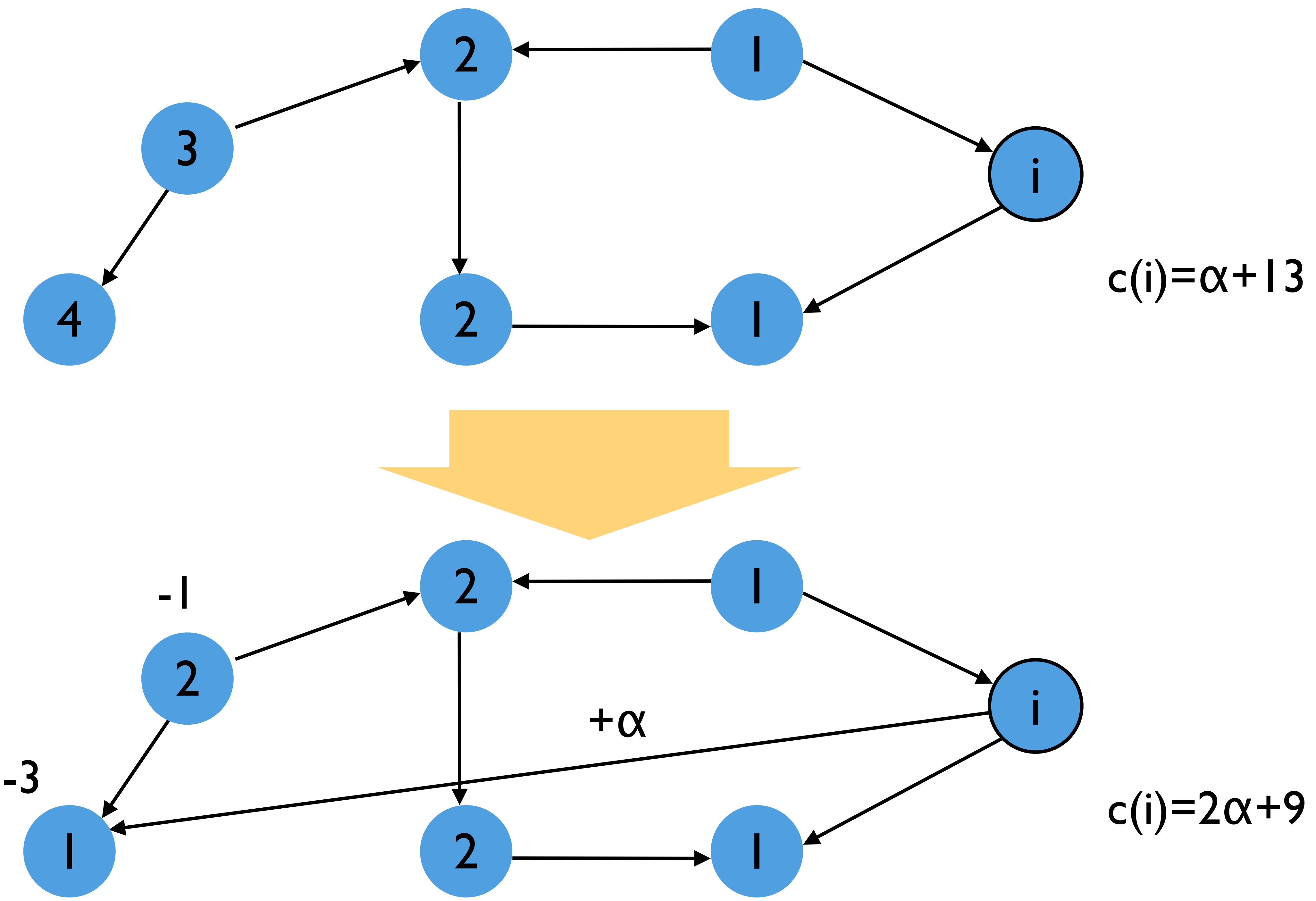
  $\alpha \cdot \#edges + \Sigma_{u,v}\, d(u,v)$

Variants of this problem

- MAXCOST
  replace $\Sigma_v\, d(u,v)$ by $\max_v d(u,v)$
- BILATERAL
  cost of edge (u,v) is charged α/2 to each vertex u,v
- SWAP EQUILIBRIUM
  graph is in equilibrium if no player u wants to change a single arc (u,v) to some arc (u,w)

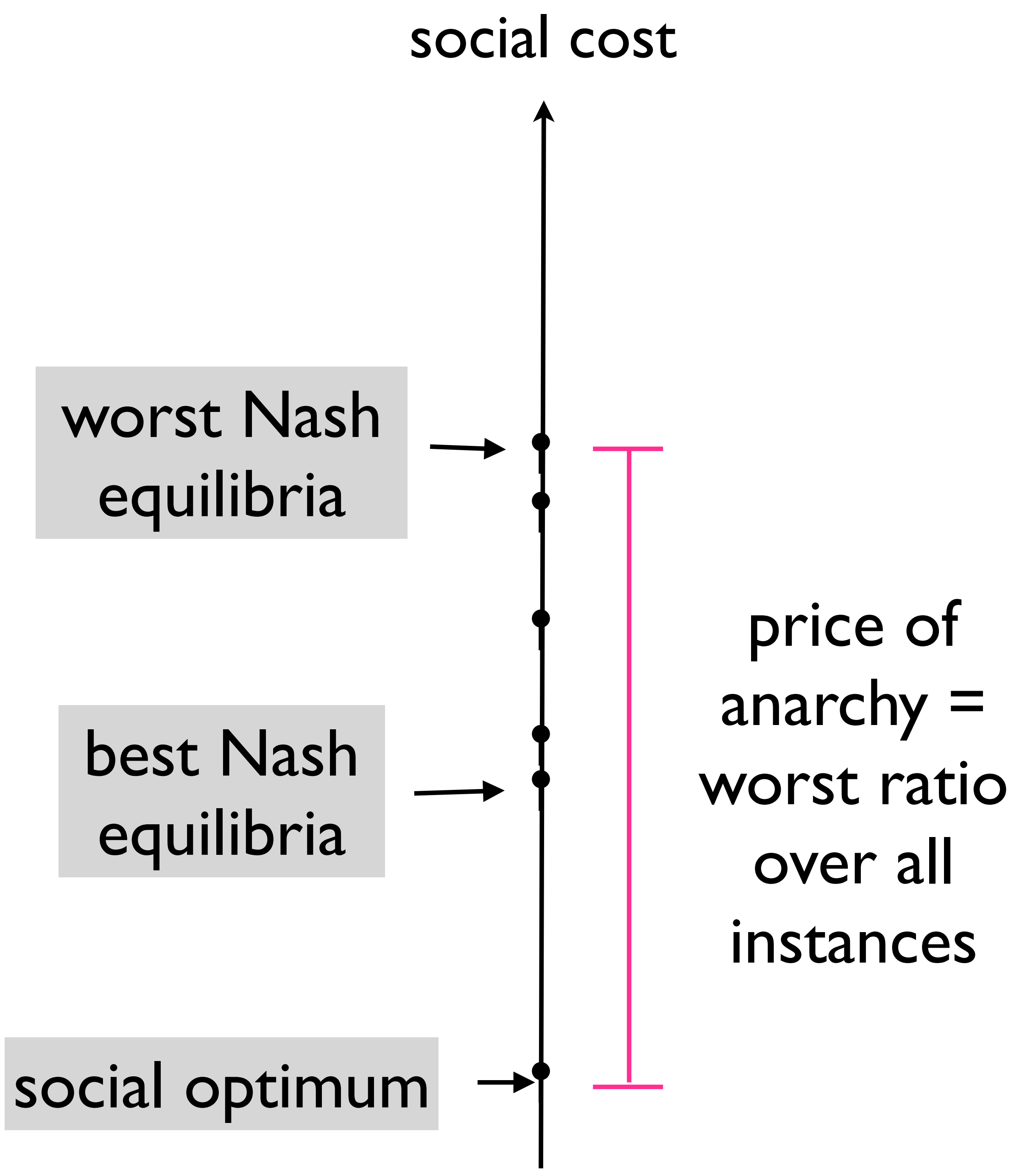
  — motivated by NP-hardness of best response in our game

Example



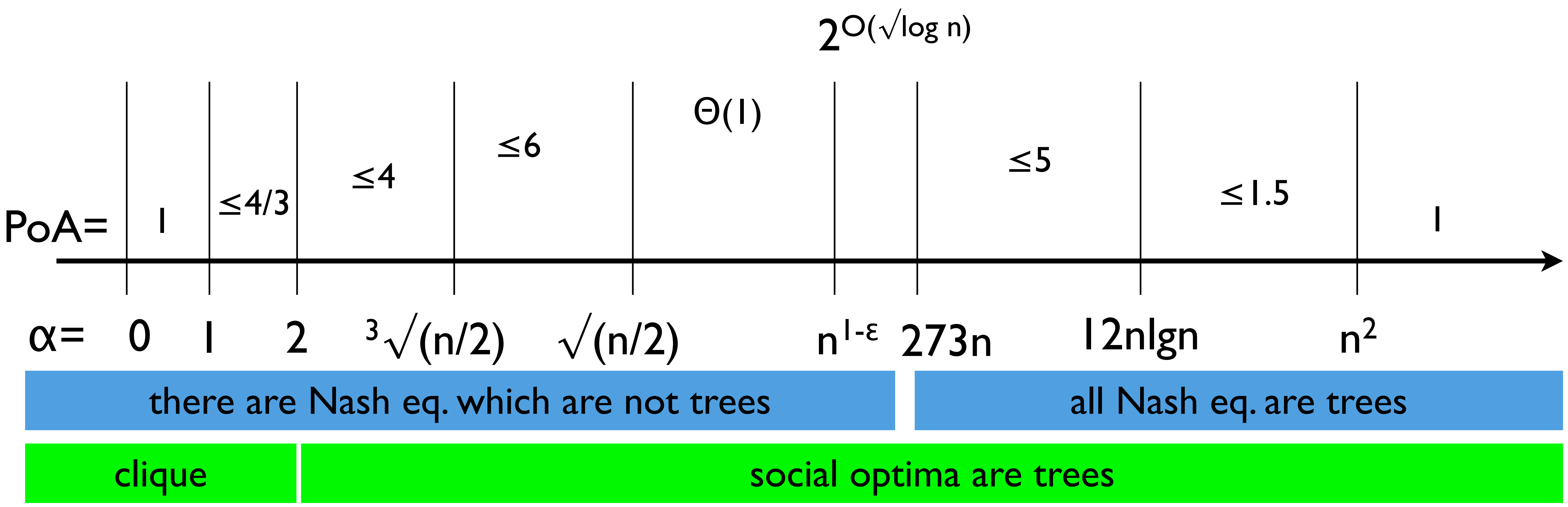
The price of anarchy

- Quality of a network = social cost
- here : sum of user costs
- the price of anarchy is a function
  of α and n

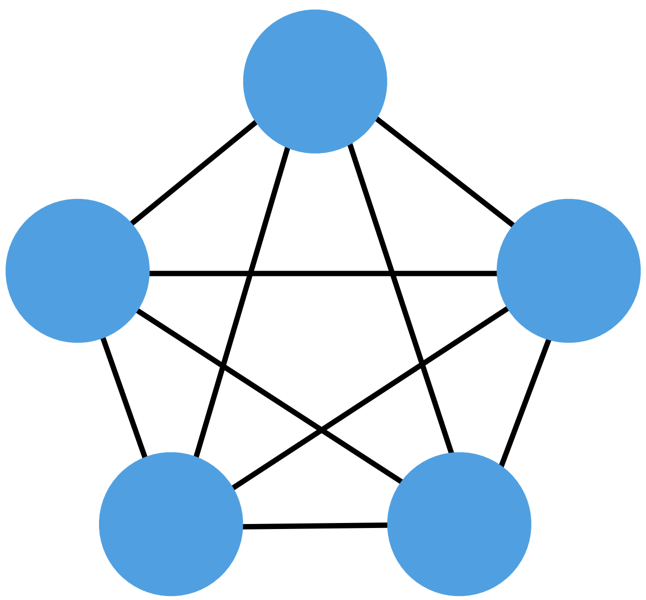


Known bounds

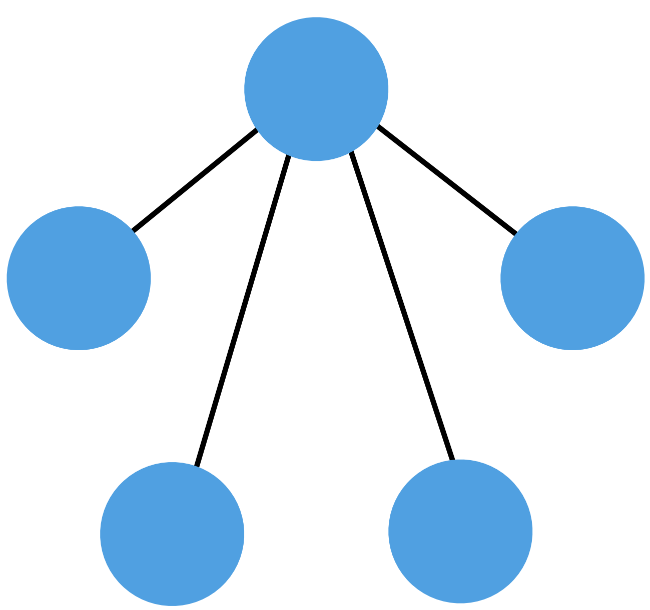
- [Fabrikant,Luthra,Maneva,Papadimitriou,Shenker'2003]
- [Lin'2003]
- [Albers,Eilts,Even-Dar,Mansour,Roditty'2006]
- [Demaine,Hajiaghayi,Mahini,Zadimgoghaddam'2007]
- [Mihalák,Schlegel'2010]



Social optima



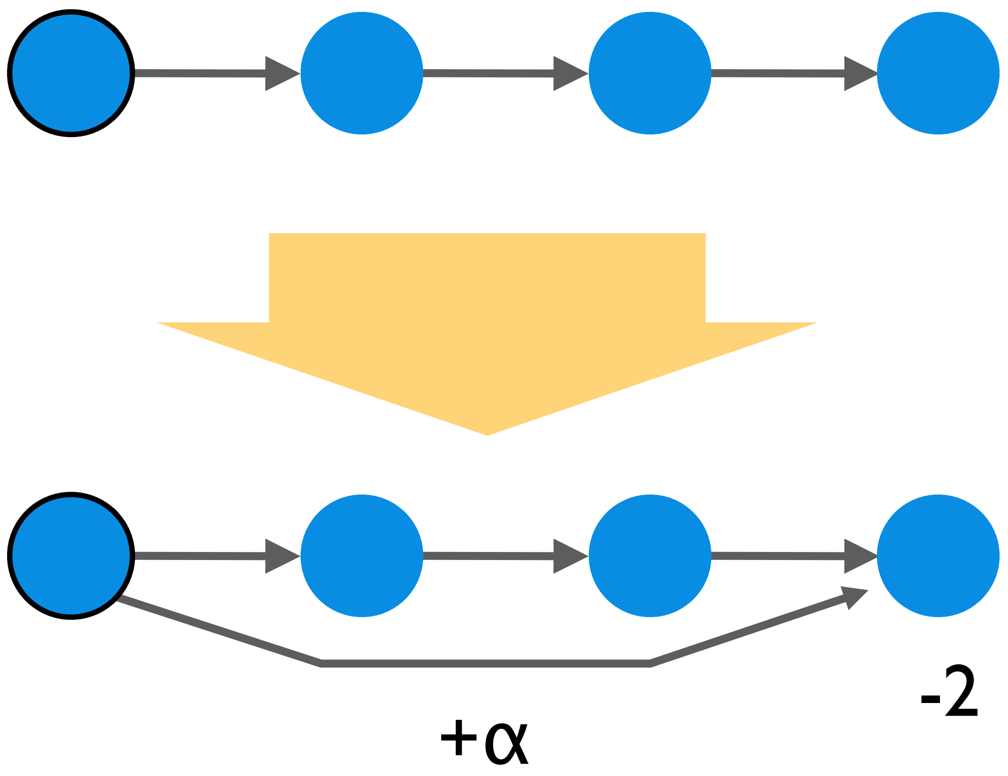
- For $\alpha<2$, social optimum is the clique: otherwise adding an edge costs α but saves 2 at least



- For $\alpha\geq 2$, social optimum is a star: any additional edge would cost too much

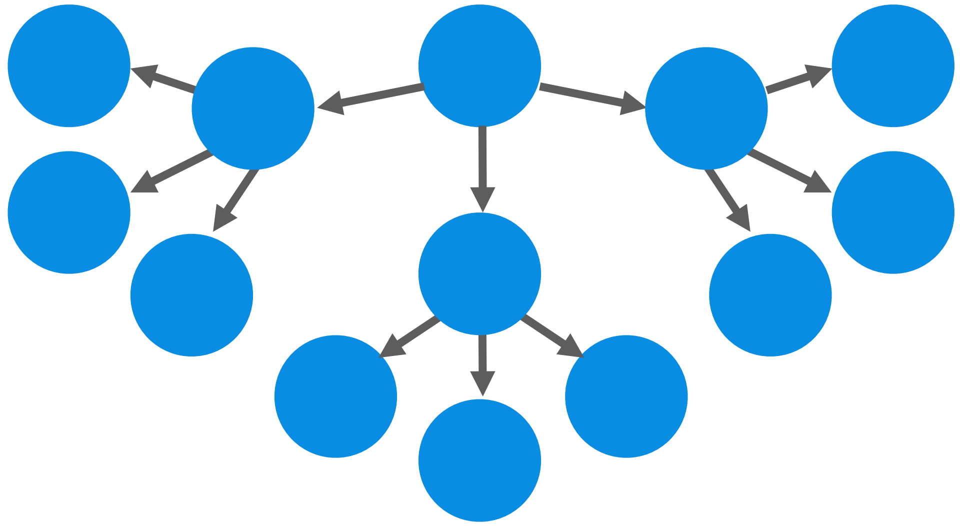
Nash equilibria

- For $\alpha<1$, the clique
- For $1<\alpha<2$, the diameter is at most 2, otherwise...
- For $1<\alpha<2$, the star is the worst Nash eq. and has price of anarchy $\leq 4/3$



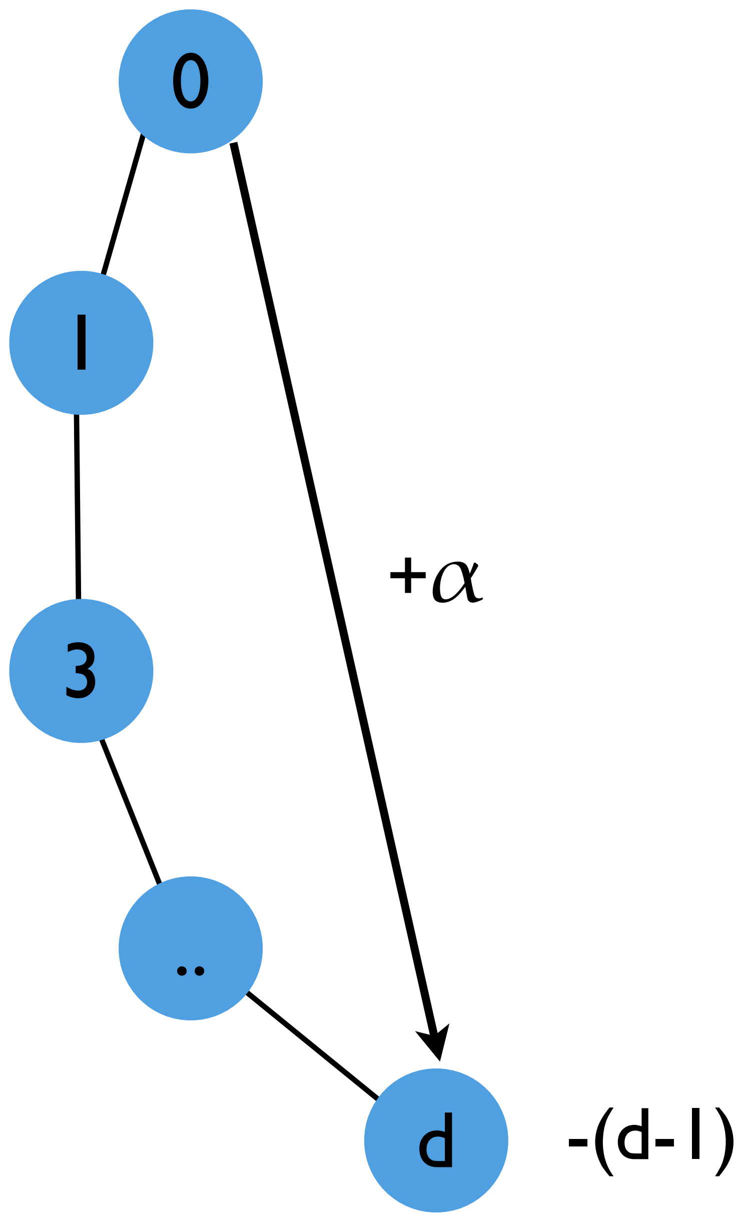
In the worst case price of anarchy is ≥3

- An outward-directed complete k-ary tree of depth d, at $\alpha=(d-1)n$:
- For large d, k, the price of anarchy approaches 3 asymptotically



When Nash eq. are trees the price of anarchy is ≤5

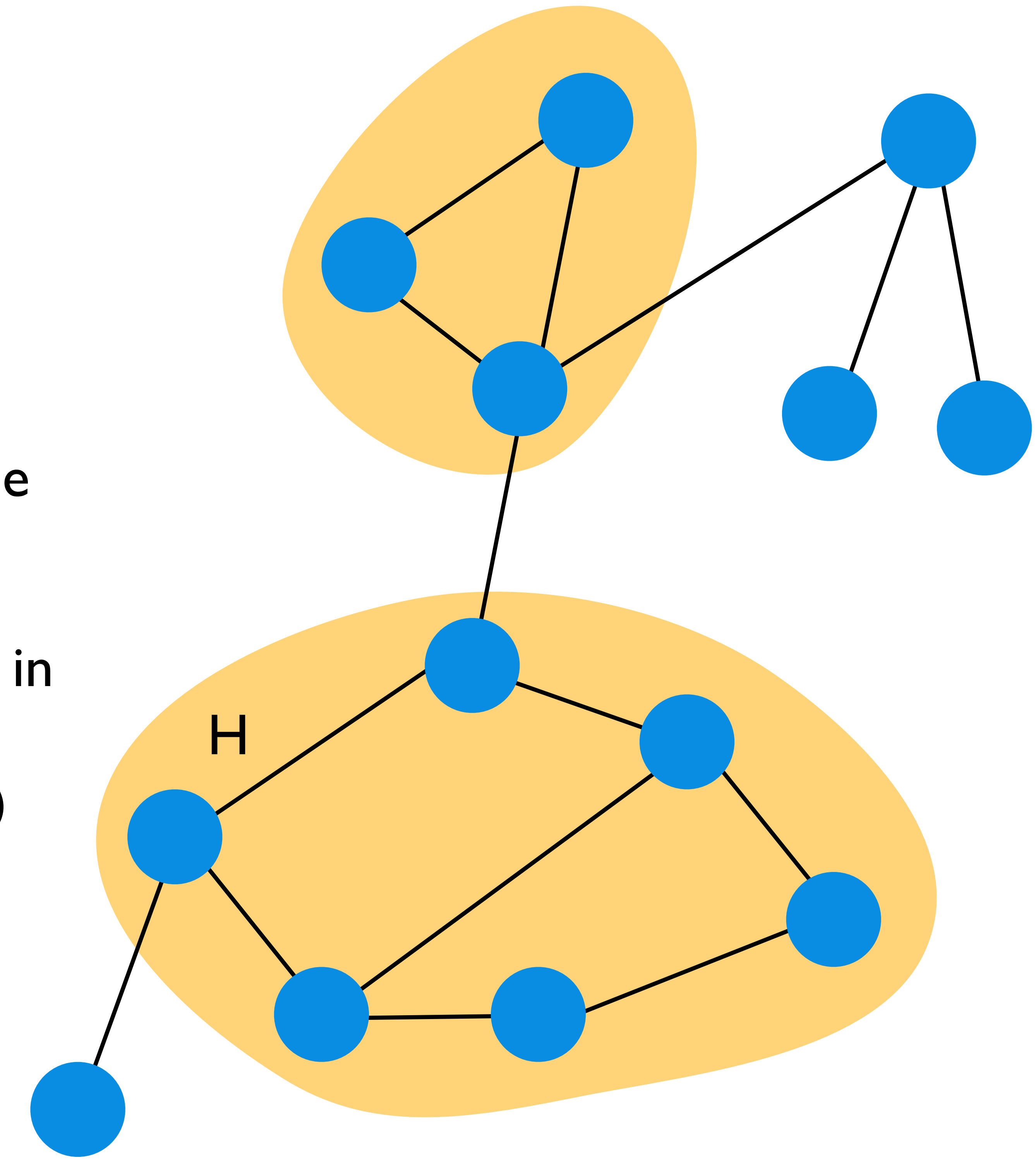
- Let a Nash equilibria be a tree of diameter d
- $\alpha > d-1$, otherwise it is not an equilibria
- Its cost is $O(\alpha n + dn) = O(\alpha n)$
- The optimal cost (star) is also $O(\alpha n)$



When α>273n all equilibria graphs are trees

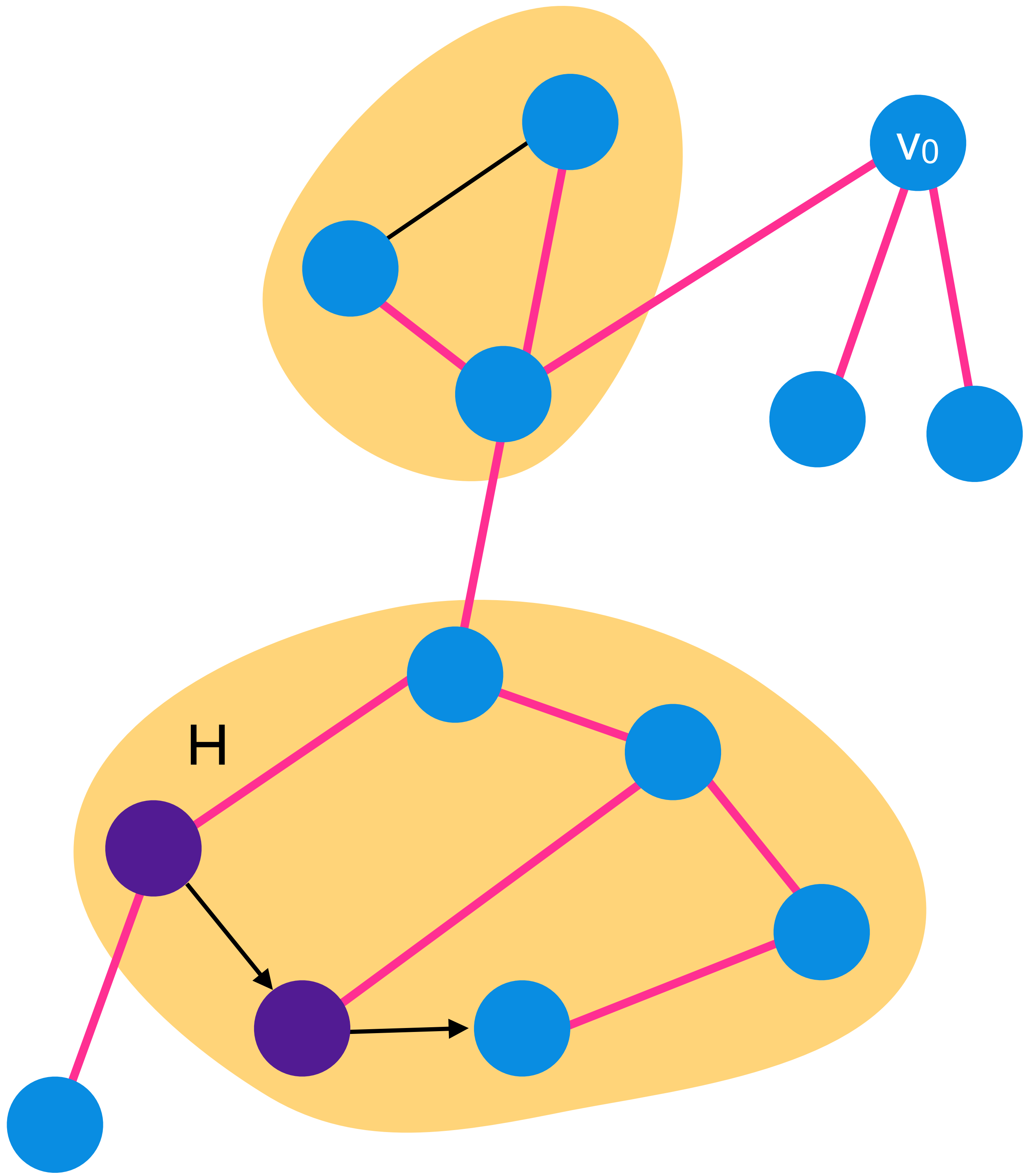
Outline of the proof

- Fix an equilibria graph
- Consider maximal bi-connected components H bigger than a single vertex
- bound average degree of vertices in H by
  $2+1/34 \leq deg(H) \leq 2+8n/(\alpha-n)$
- So we have a contradiction for α>273n



Upper bound average degree deg(H)

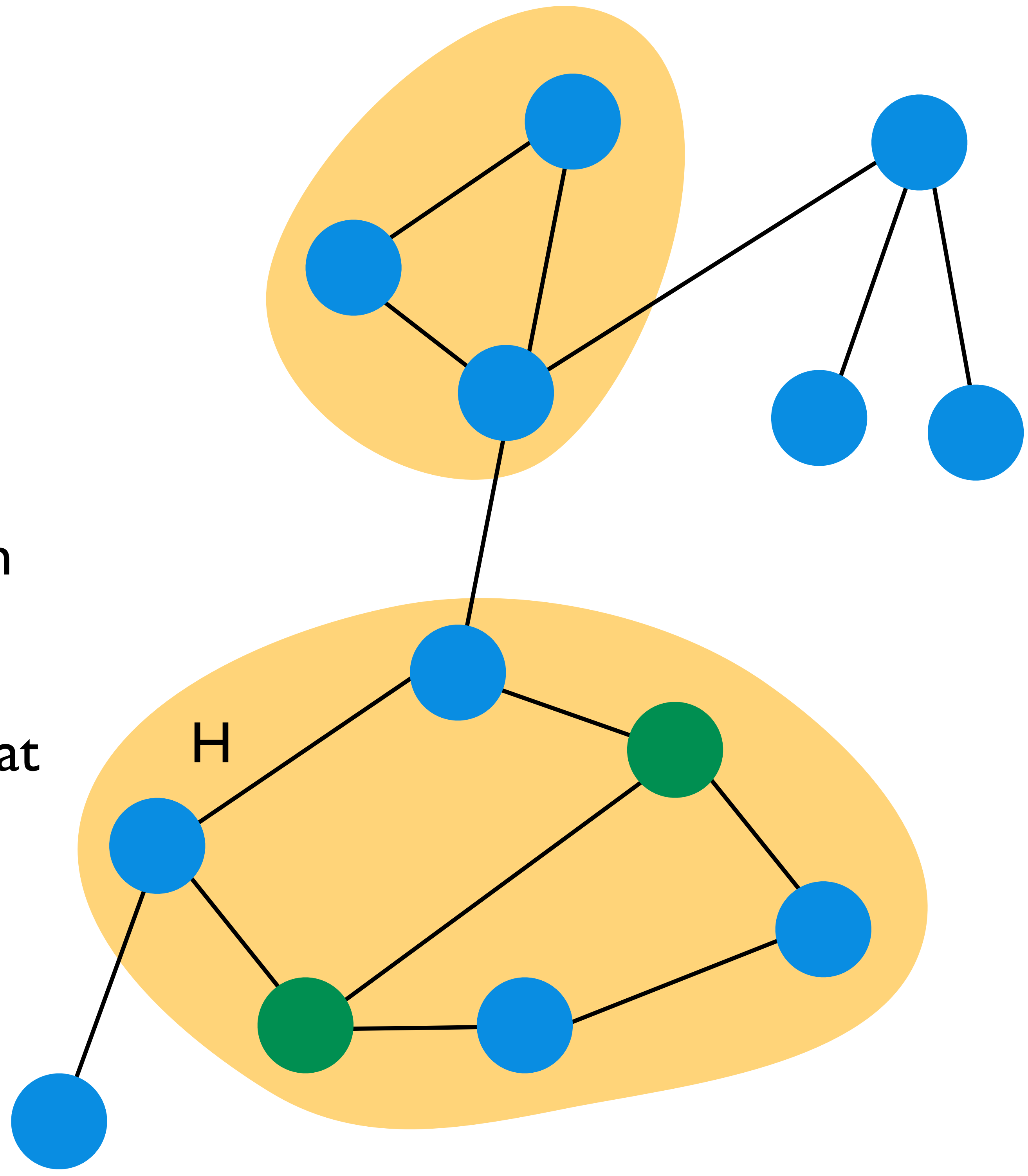
- Let v_0 be a vertex minimizing connection cost $\Sigma_u d(v_0,u)$
- Consider BFS-tree (shortest path tree) T rooted in v_0
- Edges in H partition into those on T and those not on T
- To bound deg(H) it suffices to bound #E(H\T)
- The *shopping vertices* that bought arcs from #E(H\T), bought only one. Otherwise changing to v_0 is cheaper
- The trick is to lower bound distance between two *shopping vertices*



Lower bound average degree deg(H)

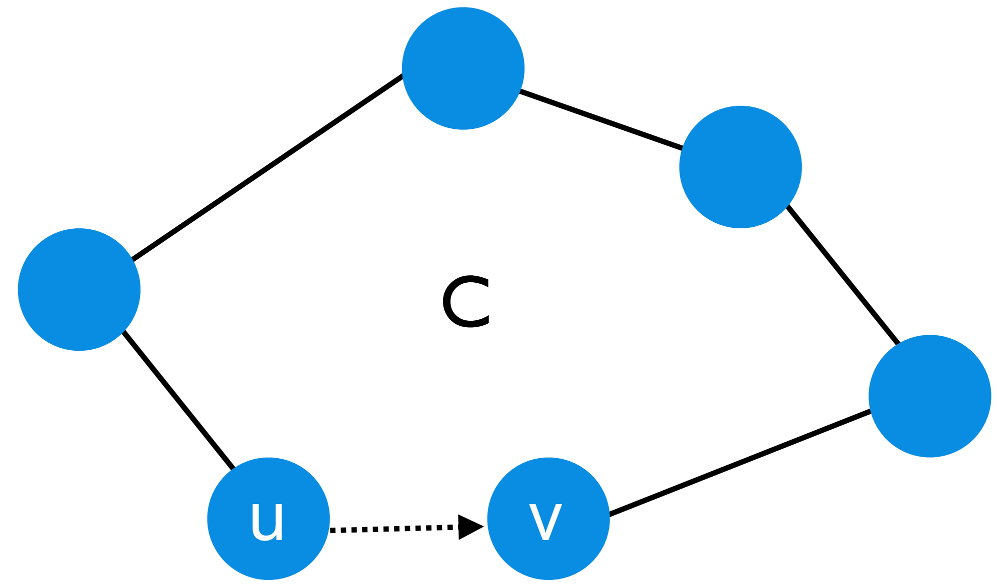
Outline of the proof

- Show that every vertex $u \in H$ is at distance at most 11 from another vertex $v \in H$ with $deg_H(v) \geq 3$, which is the degree of v restricted to H
- For this we need some lemmas that ensure existance of degree 3 vertices…

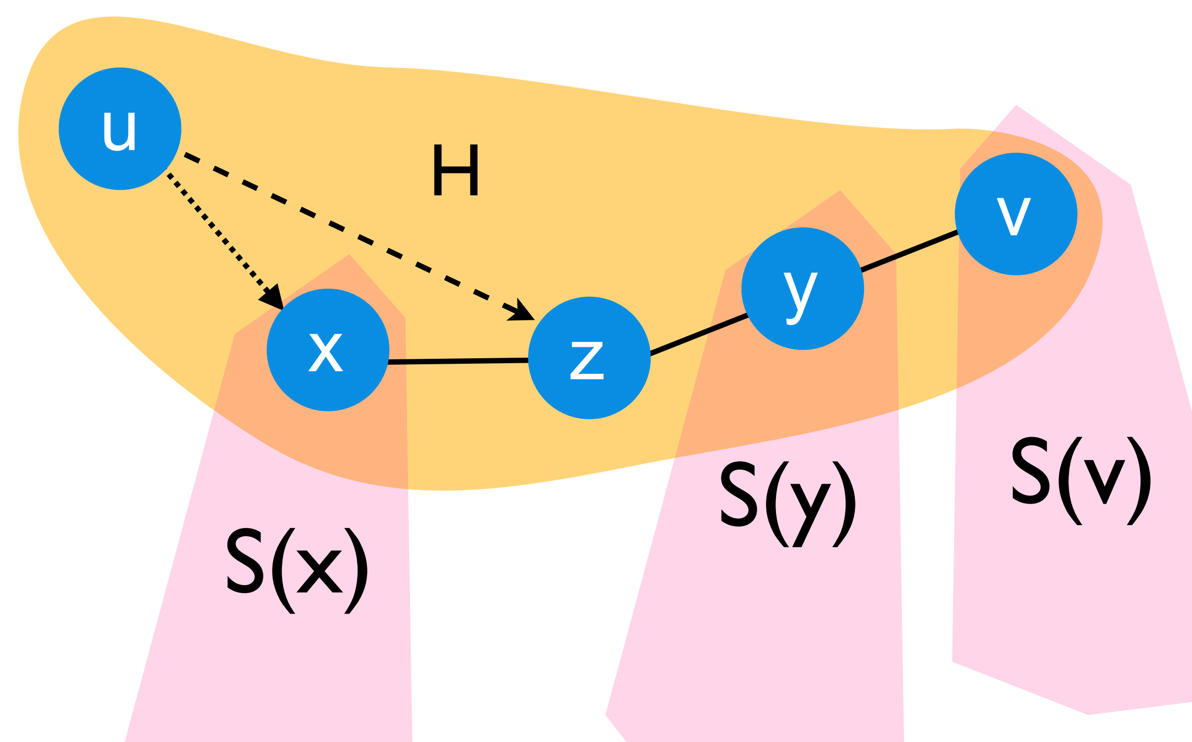


Structural lemmas

- No cycle is shorter than $\alpha/n+2$
  otherwise removing arc (u,v) from this cycle increases cost by $\leq n(\#C-1) < \alpha$



- For a length$\geq$3 shortest path in H
  $u \rightarrow x - z - \ldots - y \leftarrow v$
  one of x,y has $\deg_H \geq 3$
  otherwise changing (u,x) to (u,z) decreases distance to S(y),S(v) while increasing distance of S(x). By symmetry we assumed $\#S(x) \leq \#S(y)$



- more lemmas are needed…

S(x):=set of vertices w who's shortest path to H ends at x
Clearly $S(x) \cap H = \{x\}$, and $\{S(x)\}_{x \in H}$ partitions the graph

Summary

- network congestion game
- price of anarchy
- techniques for analyzing price of anarchy

Game Theory and Applications

4– Mechanism design

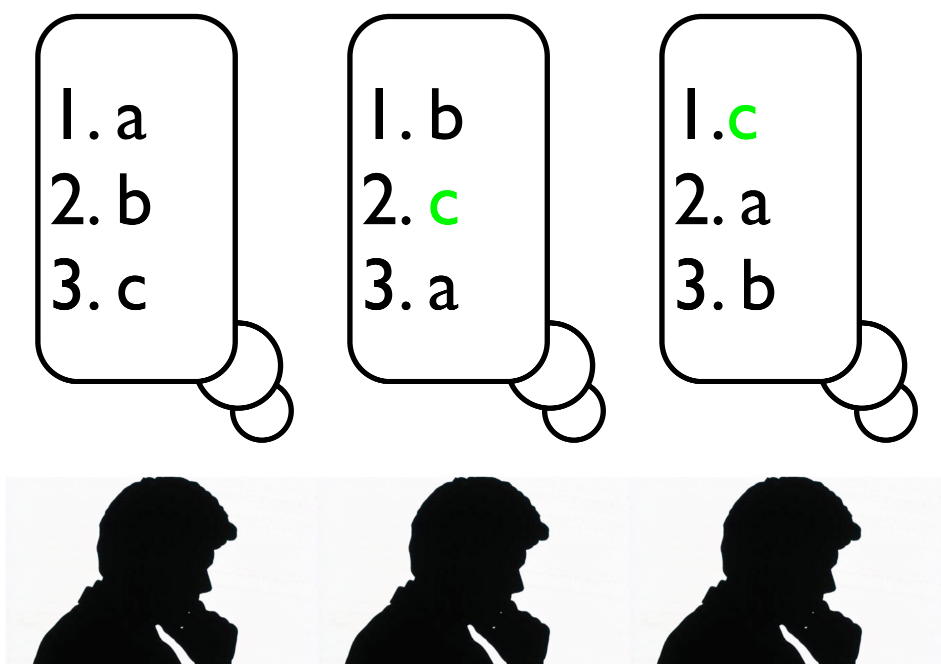
November 2012, Departamento de Ingeniería Industrial, Univ. of Santiago, Chile

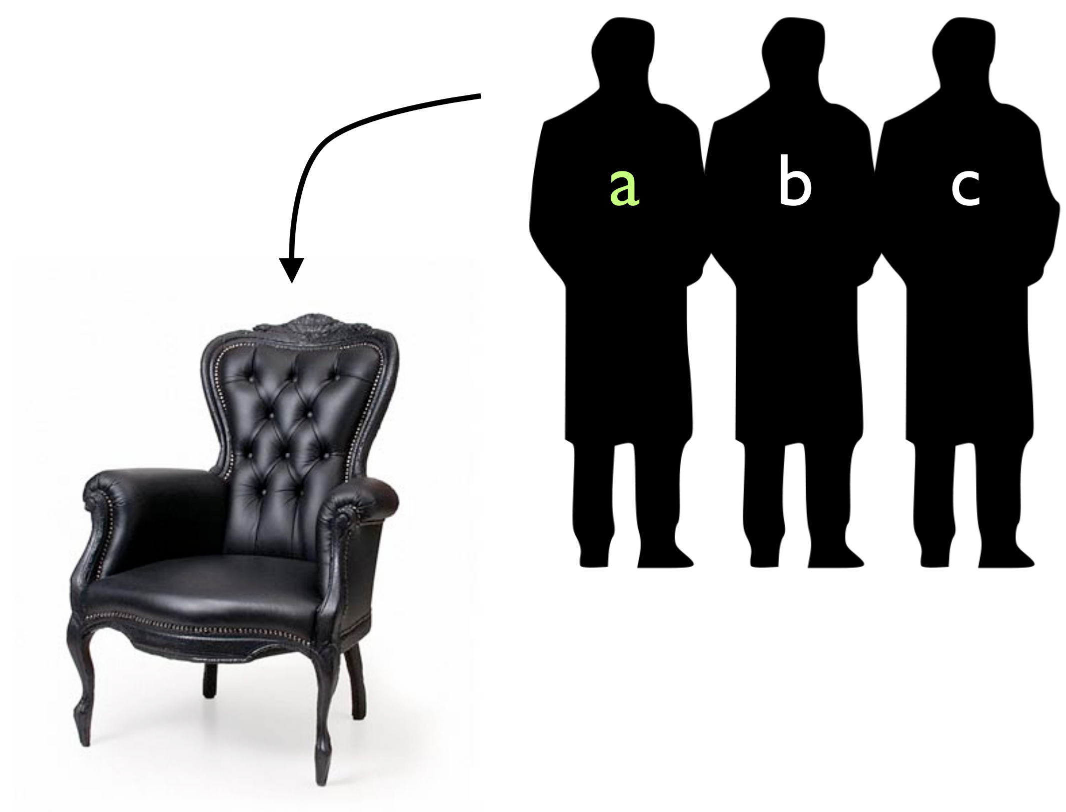
Christoph Dürr

Condorcet’s paradox

- 3 candidats for one seat
- 3 voters with different preferences
- for every outcome, always 2/3 of the voters would have prefered another candidate

- Problem: what properties should a voting procedure satisfy?





Formally (sorry)

A : set of alternatives (candidates)

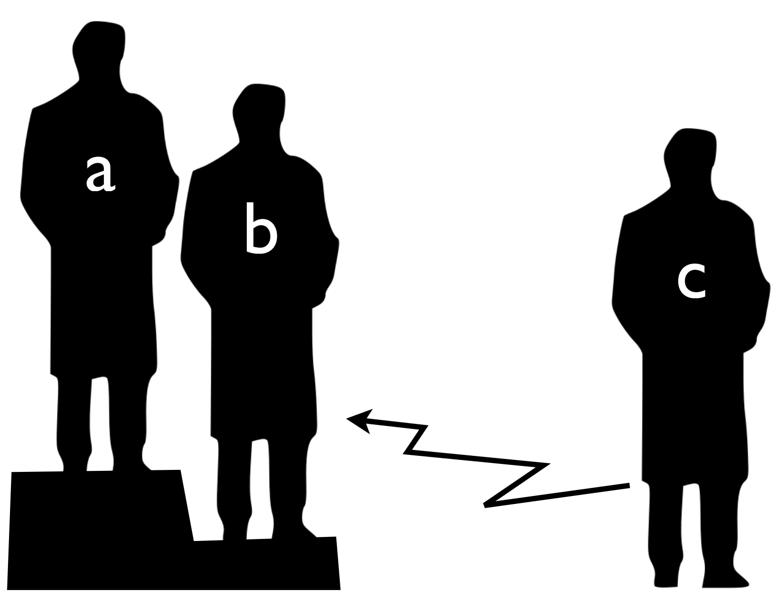
L : set of total orders on A,
$|L|=|A|!$

$>_i \in$ **L** : preferences of voter

$\pi \in$ **L**^{n} : strategy profile

F : L$^n \rightarrow$ L : social welfare function

f : L$^n \rightarrow$ A : social choice function



desirable properties

F satisfies **l'unanimity** if $F(<,\ldots,<)=<$

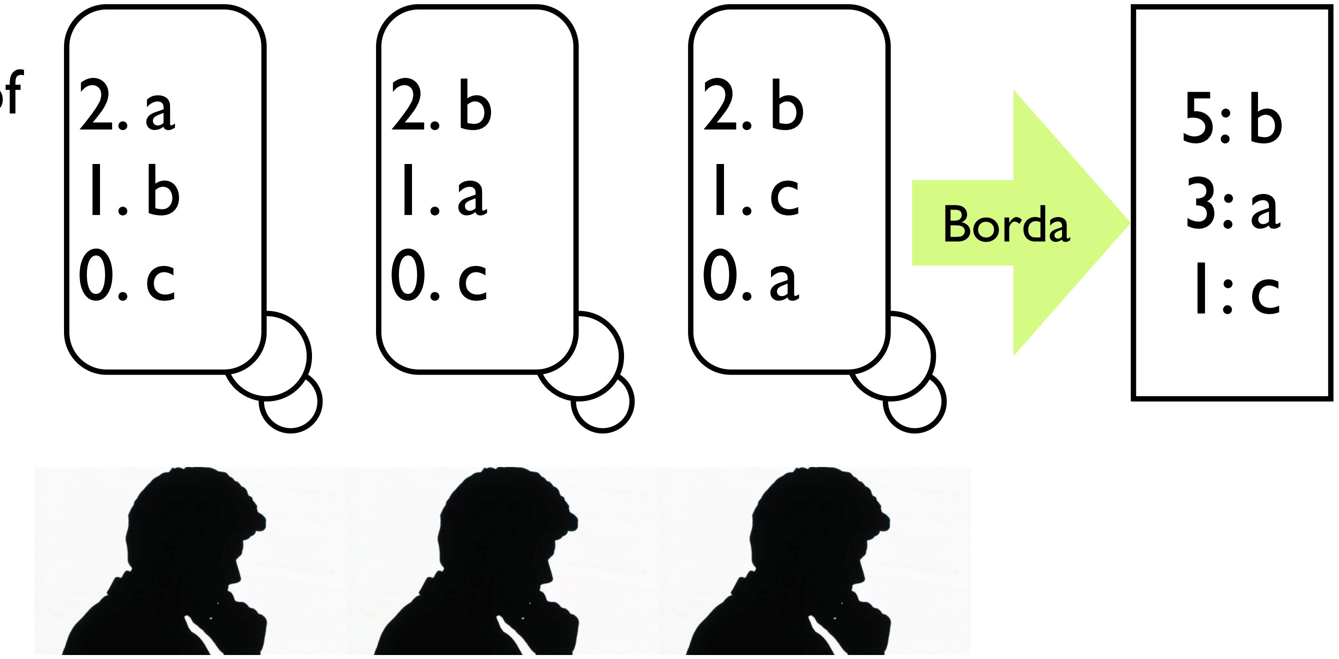
voter i is a **dictateur** if
$\forall <_1,\ldots,<_n \in L^n : F(<_1,\ldots,<_n)=<_i$

F is **independent to irrelevant alternatives** if $\forall a,b \in A$, $<_1,\ldots,<_n \in L^n$,
$<'_1,\ldots,<'_n \in L^n$ with
$\forall i$: $a <_i b$ □ $a <'_i b$
then $a < b$ □ $a <' b$ for
$<=F(<_1,\ldots,<_n)$ and $<'=F(<'_1,\ldots,<'_n)$

Borda's rule

[100 and 1770]

- Every voter ranks the alternatives. The score of alternative is the sum of ranks. We order them by score.
- This vote satisfies the unanimity, it is dictator-free, but dépendant to irrelevant alternatives
- proof by example :



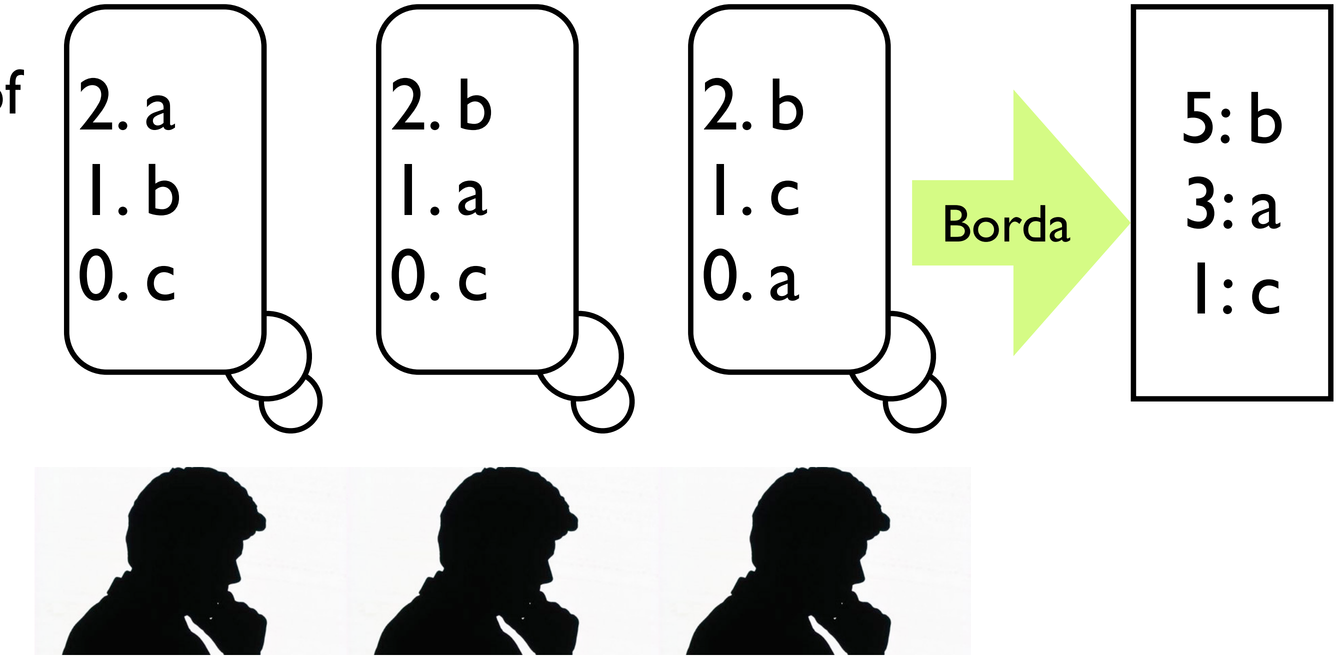
exercise:

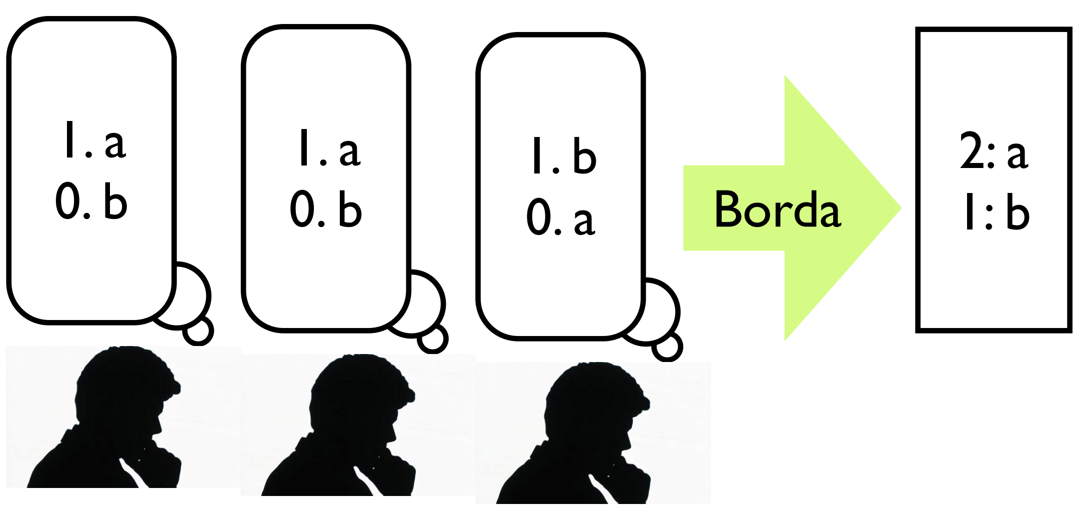
Give an example were Borda's rule is dependent to irrelevant alternatives

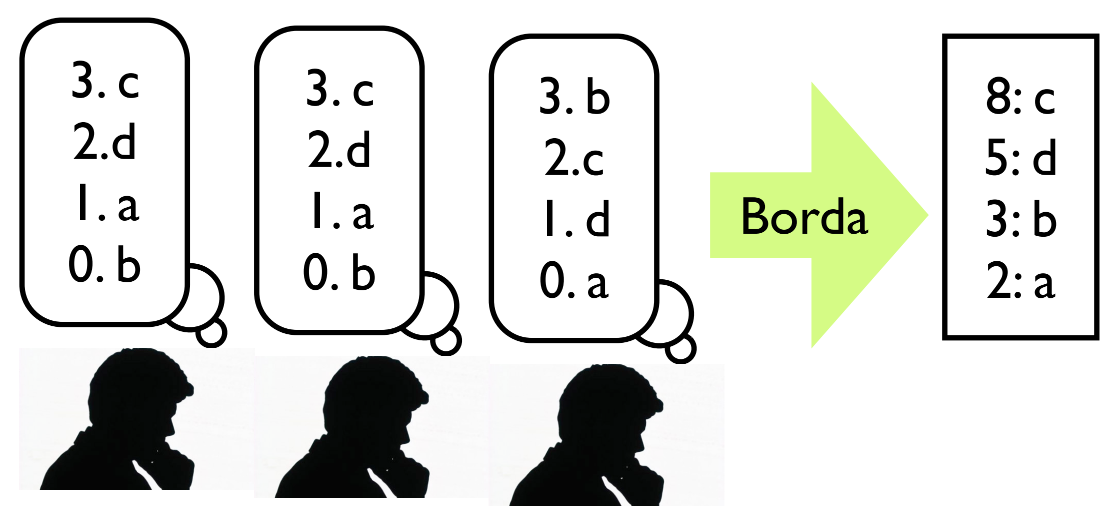
Borda’s rule

[100 and 1770]

- Every voter ranks the alternatives. The score of alternative is the sum of ranks. We order them by score.
- This vote satisfies the unanimity, it is dictator-free, but dépendant to irrelevant alternatives
- proof by example :







Arrow's impossibility Thm

1951

- ∀A : |A|≥3, ∀ socal welfare function F such that

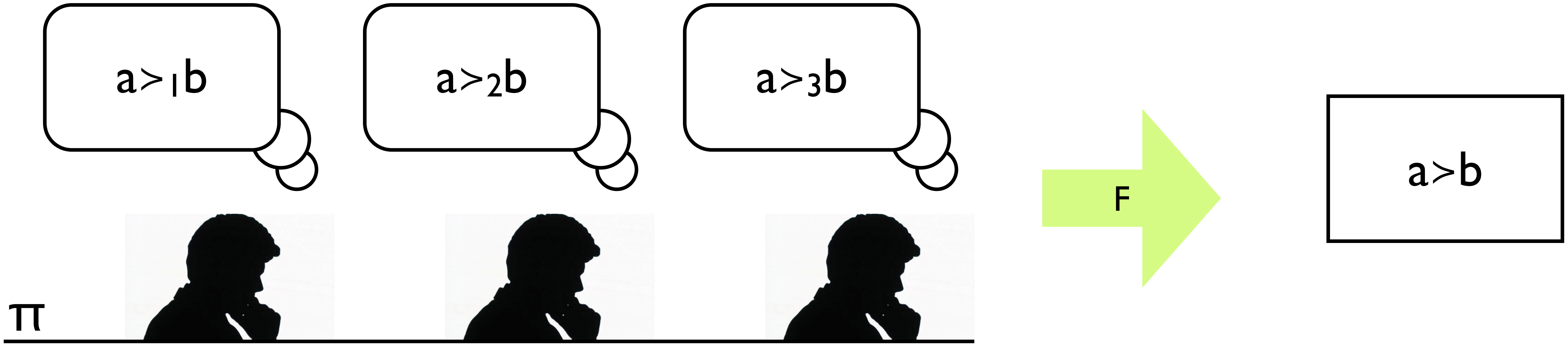
unanimity independance to irrelevant alternatives (iia)	⇒	there is a dictator

Many proofs have been proposed, here is one by Geanakoplos …
2005

- from now on, let be A,F with |A|≥3 and F a function satisfying the unanimity and iia

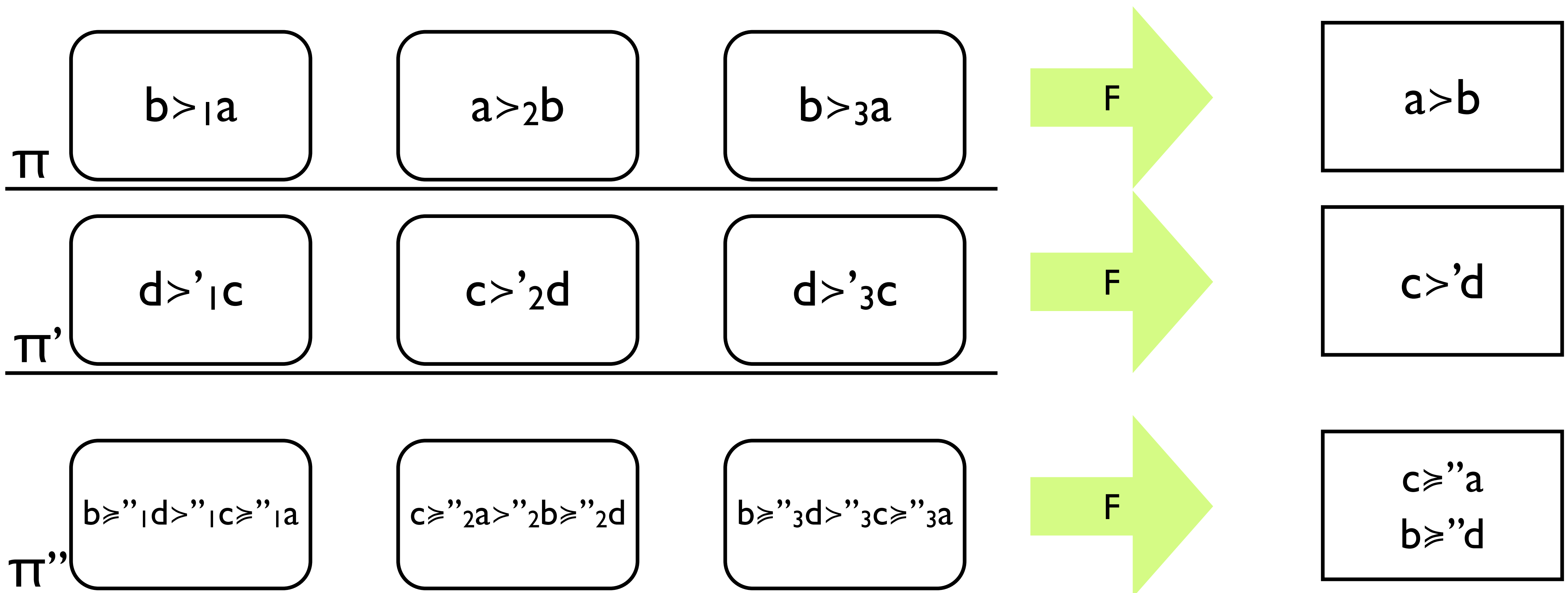
Unanimity + iia

- **Prop** If all voters prefer a to b, then so must the outcome.
- **proof**: given strategy profile $\pi=(>_1,\ldots,>_n)$ with $\forall i{:}a>_i b$ we construct the profile $\pi'=(>_1,\ldots,>_1)$.
- By unanimity $a>'b$ for $>'=F(\pi')$
- By iia, $a>b$ pour $>=F(\pi)$
- $\square$



Strict Neutrality

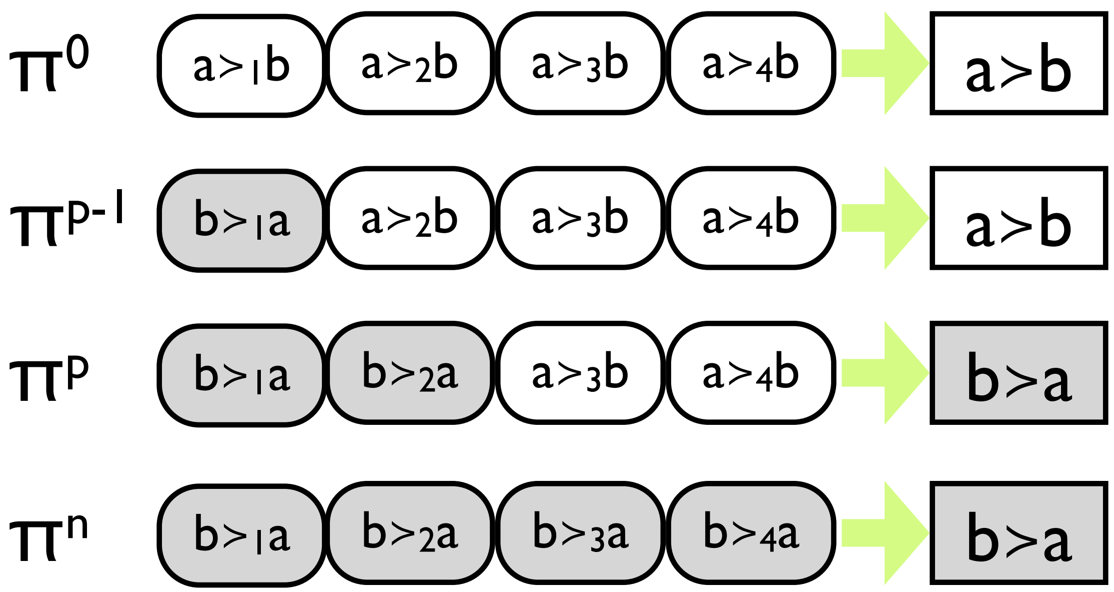
- **Prop** Let be a,b,c,d∈A, a≠b, c≠d. If all voters compare ab the same way as cd, then so must the outcome.
- **proof** Wlog a>b. We construct another profile π'' compatible with π on ab and compatible with π' on cd and where everyone prefers c to a et b to d (except if c=a ou b=d)
- By unamity c≽''a et b≽''d
- By iia a>''b
- By transitivity c>''d
- By iia c>'d □



identify a dictator

- Let be $a,b \in A$, and for all $i=0\ldots n$ be profiles $\pi^i=(>_1,\ldots,>_n) \in L^n$ where $b >_1 a, \ldots\ b >_i a,\ a >_{i+1} b, \ldots, a >_n b$

- By unanimity $F(\pi^0)$ prefers a to b however $F(\pi^n)$ prefers b to a

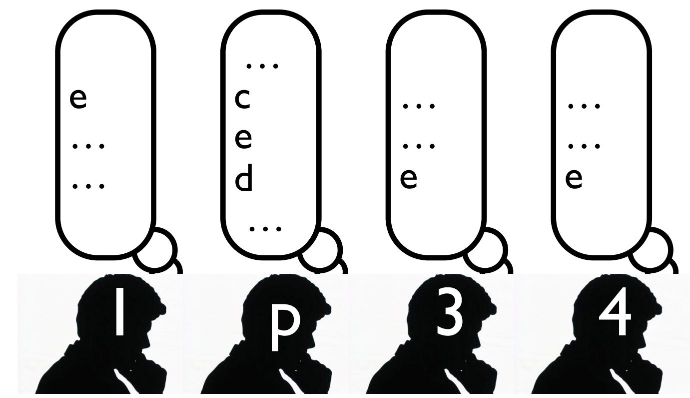


- Let p be the first voter where $F(\pi^p)$ prefers b to a

- **Prop** p is a dictator

- **proof** Let $\pi= (>_1,\ldots,>_n)$ be an arbitrary profile and $c,d \in A$ such that $c >_p d$.
  Let be $>=F(\pi)$.

- Let be $e \in A$, $e \notin \{c,d\}$. Transform π into π' such that
  $e >'_j x \quad \forall x \in A, \forall j<p$
  $c >'_p e > d$
  $x >'_j e \quad \forall x \in A, \forall j>p$



exercise:
use neutrality to show that $c>d$.

identify a dictator

- Let be $a,b \in A$, and for all $i=0\ldots n$ be profiles $\pi^i=(>_1,\ldots,>_n) \in L^n$ where $b>_1 a,\ldots\ b>_i a,\ a>_{i+1} b,\ldots,\ a>_n b$

- By unanimity $F(\pi^0)$ prefers a to b however $F(\pi^n)$ prefers b to a

π^0	$a>_1b$	$a>_2b$	$a>_3b$	$a>_4b$	$a>b$
π^{p-1}	$b>_1a$	$a>_2b$	$a>_3b$	$a>_4b$	$a>b$
π^p	$b>_1a$	$b>_2a$	$a>_3b$	$a>_4b$	$b>a$
π^n	$b>_1a$	$b>_2a$	$b>_3a$	$b>_4a$	$b>a$

- Let p be the first voter where $F(\pi^p)$ prefers b to a

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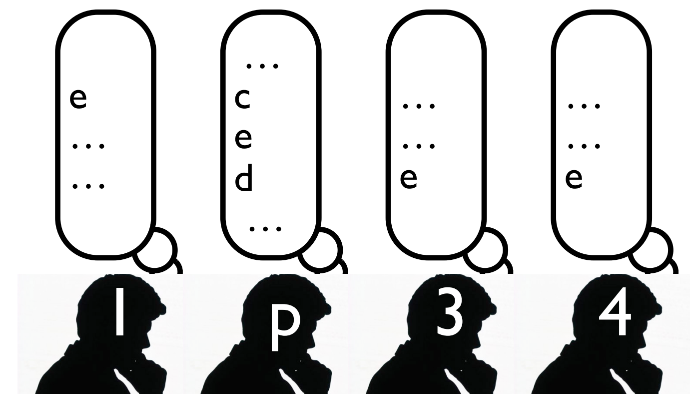


- π ou π' agree on the order cd, but $c>d$ iff $c>'d$ for $>'=F(\pi')$.

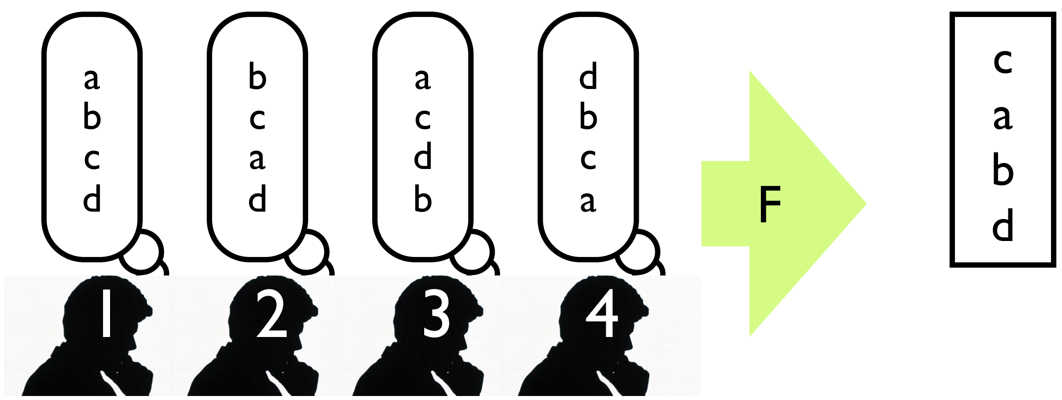
- the order of *ce* in π' = the order of *ab* in π^p, then by neutrality $c>'e$

- the order of *de* in π' = the order of *ab* in π^{p-1}, then by neutrality $e>'d$

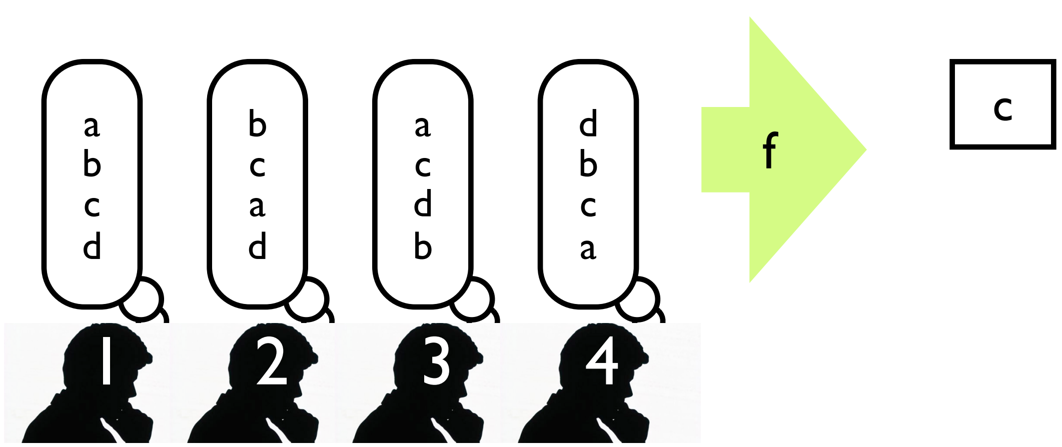
- by transitivity $c>'d$ and by iia $c>d$. □

strategic manipulations

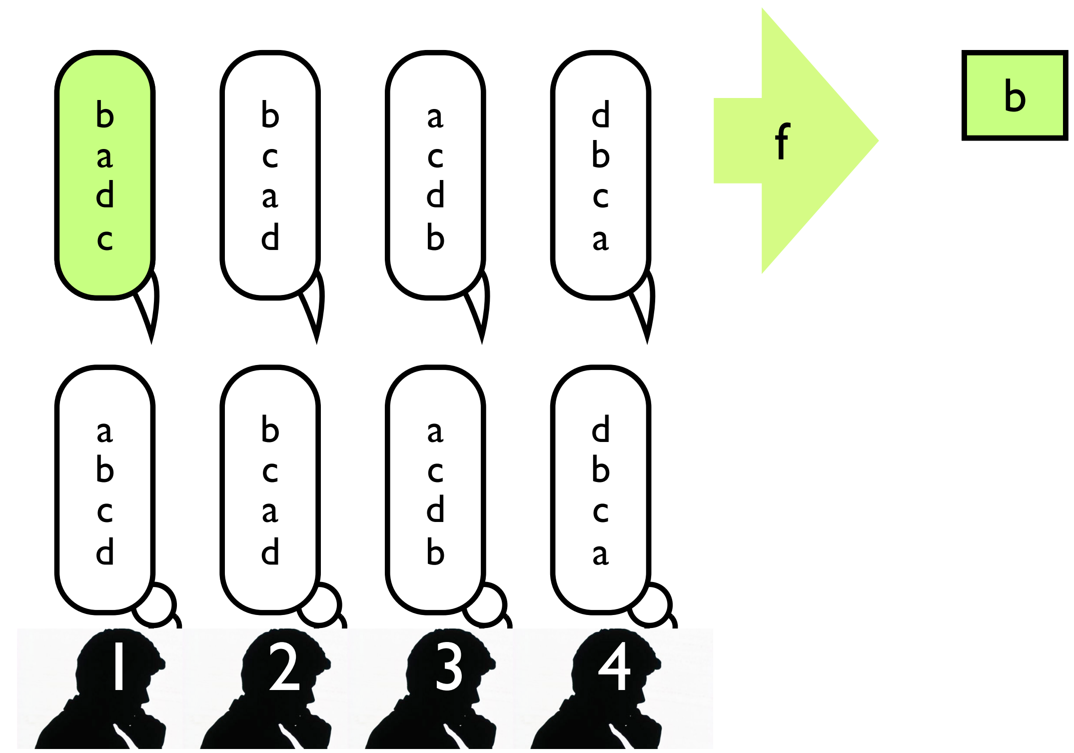
- **notation changes** (sorry)
- From now on we are not interested in the social welfare function F anymore



- only in the social choice function f



- a realistic model: voters announce preferences in public and social choice function is known to everyone.

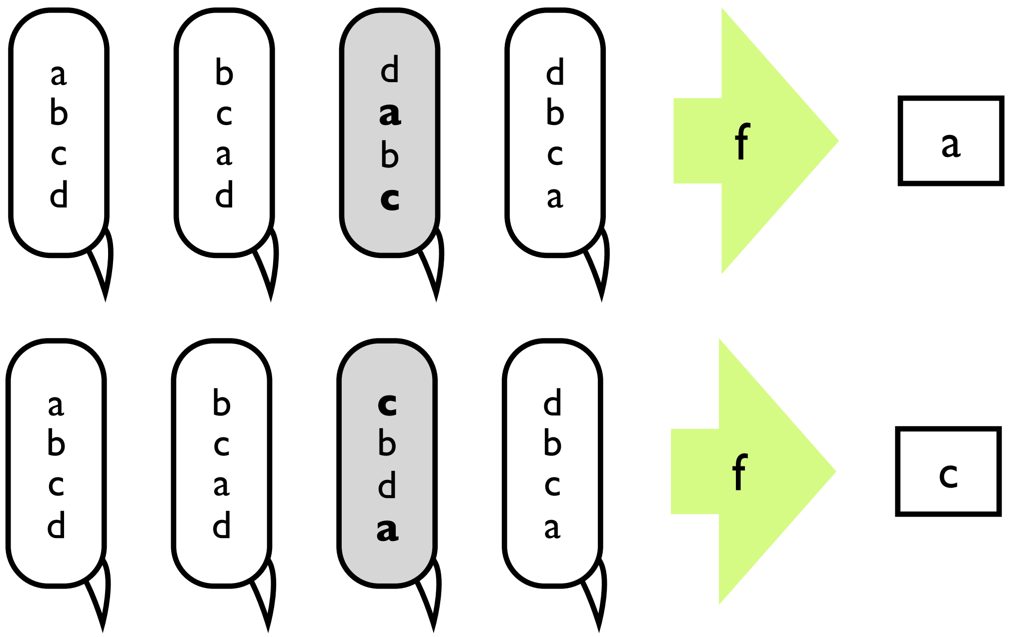


- f could be manipulated by a voter who announces something different from its preference to influence the outcome
- Other f **resists to manipulations**

equivalent names :
strategy proof,
truthful, incentive compatible

monotonicity

- **f est monotone** if $f(>_1,\ldots,>_i,\ldots,>_n)=a\neq c=f(>_1,\ldots,>'_i,\ldots,>_n)$ implies $a>_i c$ and $c>'_i a$



- **Prop** f is monontone iff f resists to manipulations

- **proof** not being monotone for $>_i$ and $>'_i$ is equivalent to (a voter with preference $>_i$ can manipulate by voting $>'_i$ OR a voter with preference $>'_i$ can manipulate by voting $>_i$) □

- i is a **dictator** for f if for all $\pi\in L^n$, $f(\pi)$ is the top-choice of i

- **Théorème de Gibbard-Sattherwaite** 1973 1975
  $\forall A$ $|A|\geq 3$, $\forall$ social choice function f

$$\frac{f: L^n \rightarrow A \text{ is surjective} \quad \text{resists to manipulations}}{\Rightarrow \text{there is a dictator}}$$

- proof (not presented) by reduction to Arrow's theorem

Introduce payments

to circumvent the impossibilities

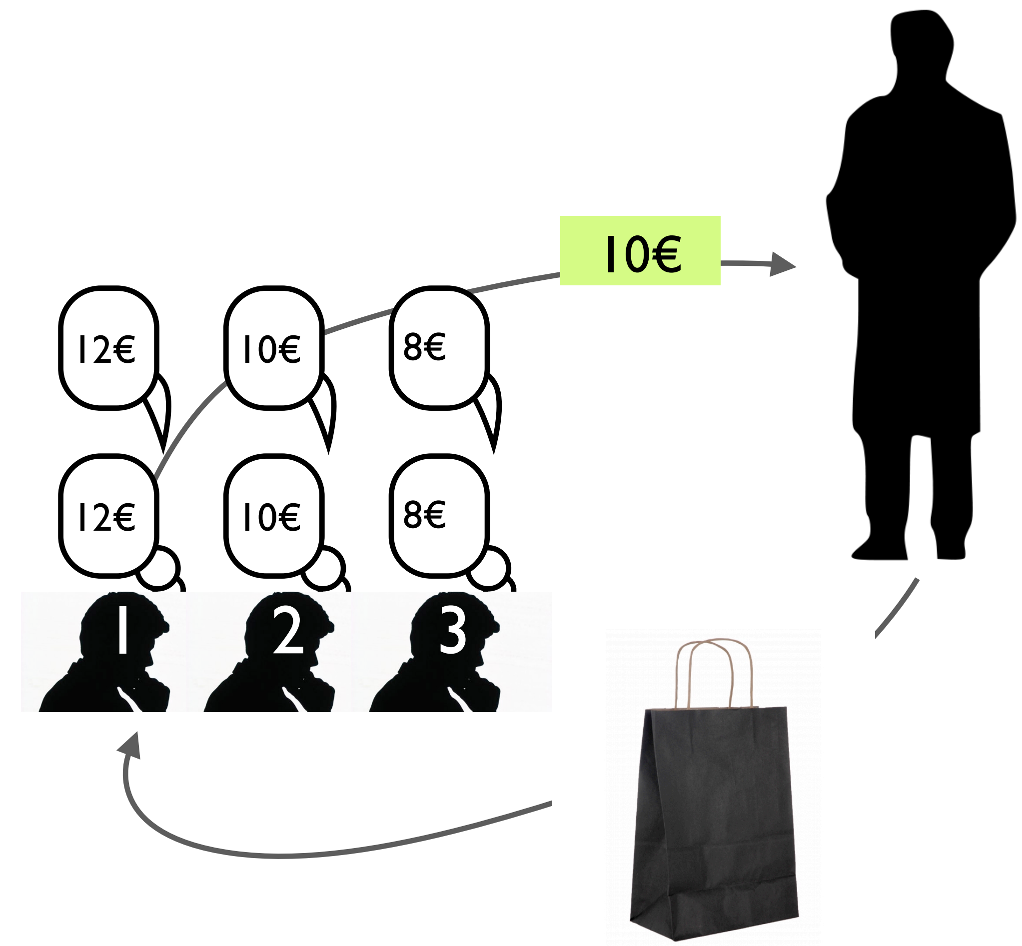
- example **second price auction for single item**
- private value $t_i \in \mathbb{R}$
- announce price $b_i \in \mathbb{R}$

player i

- social choice $k = \text{argmax } b_i$
- payment
  $p_j = 0$ for $j \neq k$
  $p_k = \max_{j \neq k} b_j$

mechanisme

- utility
  $u_j = 0$ for $j \neq k$
  $u_k = t_k - p_k$
- This mechanisme maximises the social welfare : $\sum t_i(a)$,
  where t_i(j-gagne):=t_i si i=j et :=0 sinon



exercise:
Show that this mechanism resists to manipulations.

Introduce payments

to circumvent the impossibilities

- example **second price auction for single item**
- private value $t_i \in \mathbb{R}$
- announce price $b_i \in \mathbb{R}$

player i

- social choice $k = \text{argmax } b_i$
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mechanisme

- utility
  $u_j=0$ for $j \neq k$
  $u_k=t_k-p_k$
- This mechanisme maximises the social welfare : $\sum t_i(a)$,
  where $t_i(\text{j-gagne}):=t_i$ si $i=j$ et $:=0$ sinon

- **Prop** this mechanism is resistant to manipulations
- **preuve** the idea is that the charge to the winner is independent of its bid. Bidding more does not make him win more, and bidding less, might make him loose.

  For loosers there is no reason to bid more than their value.

  □

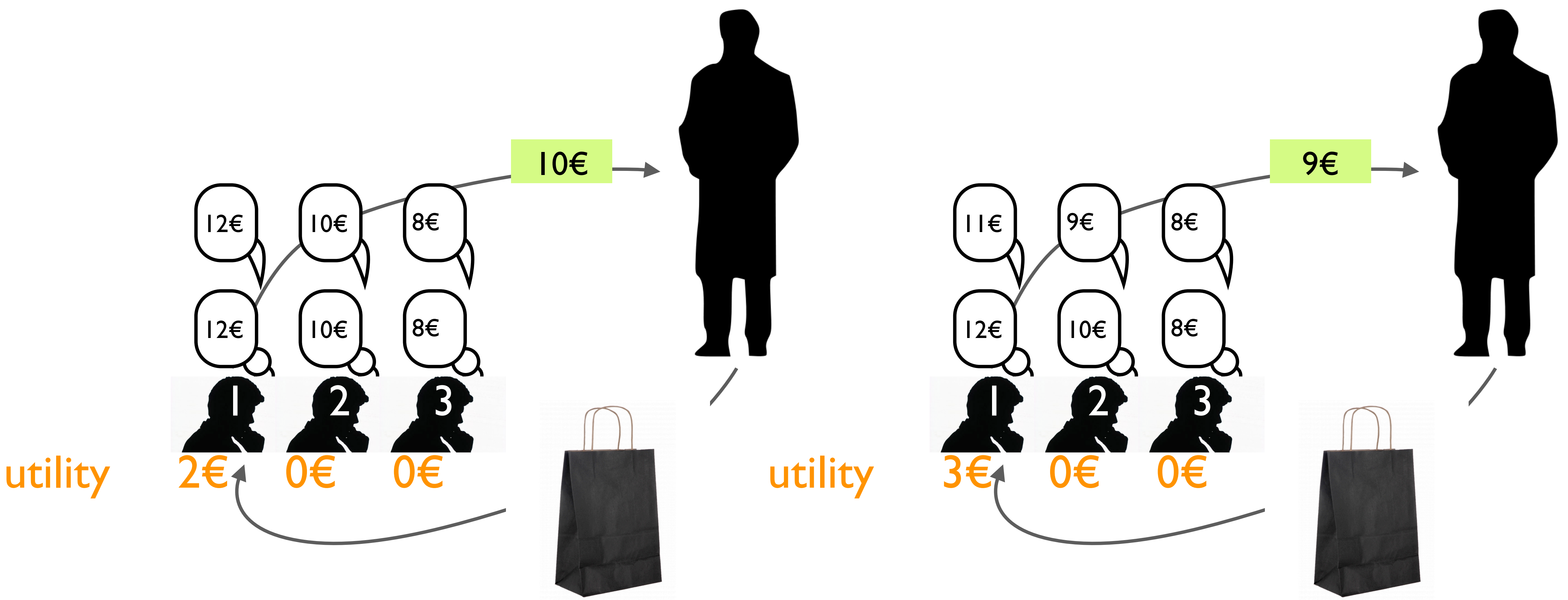
Manipulations in coallitions

- **Prop** this mechanism is not resistant to manipulations by coallitions

exercise:
Show that it is possible for players to form a coallition and agree on bids, such that no-one decreases its utility and one member of the group strictly increases its utility

Manipulations in coallitions

- **Prop** this mechanism is not resistant to manipulations by coallitions
- **proof by example**



Games with payments

notations (sorry)

A : set of alternatives

i index of player

private value $\mathbf{t_i} \in \mathbb{R}^A$

declared bid $\mathbf{b_i} \in \mathbb{R}^A$

$\mathbf{f : b \mapsto A}$: social choice function

p_i: $\mathbf{b} \mapsto \mathbb{R}$ payoff for player i

utility for player i $\mathbf{u_i}(b) := t_i(f(b)) - p_i(b)$

Example 2nd price auction

$t_i(\textit{j-wins}) = 0$ pour $i \neq j$

$b_i(\textit{j-wins}) = 0$ pour $i \neq j$

$f = \textit{i-wins}$ pour $b_i \geq b_j \ \forall j$

$p_i = \max_{j \neq i} b_j$

Vickrey Clarke Groves

1961 1971 1973

- in general:
  a mechanism $f,(p_i)$ is **VCG** if
- $f(b) \in \text{argmax}_{a \in A} \sum_i b_i(a)$
- $p_i(b) = h_i(b_{-i}) - \sum_{j \neq i} b_j(f(b))$
  for functions (h_i)
- **Theorem** if a mechanisme is VCG then it resits to manipulations
- **proof** we have to show that the utility u'_i obtained by declaring t_i is not less than by declaring $b_i \neq t_i$
- Let be $a=f(b)$, $a'=f(b_{-i},t_i)$
  $u_i = t_i(a) + \sum_{j \neq i} b_i(a) - h_i(b_{-i})$
  $u'_i = t_i(a') + \sum_{j \neq i} b_i(a') - h_i(b_{-i})$
- but this part is maximized by a'.

□

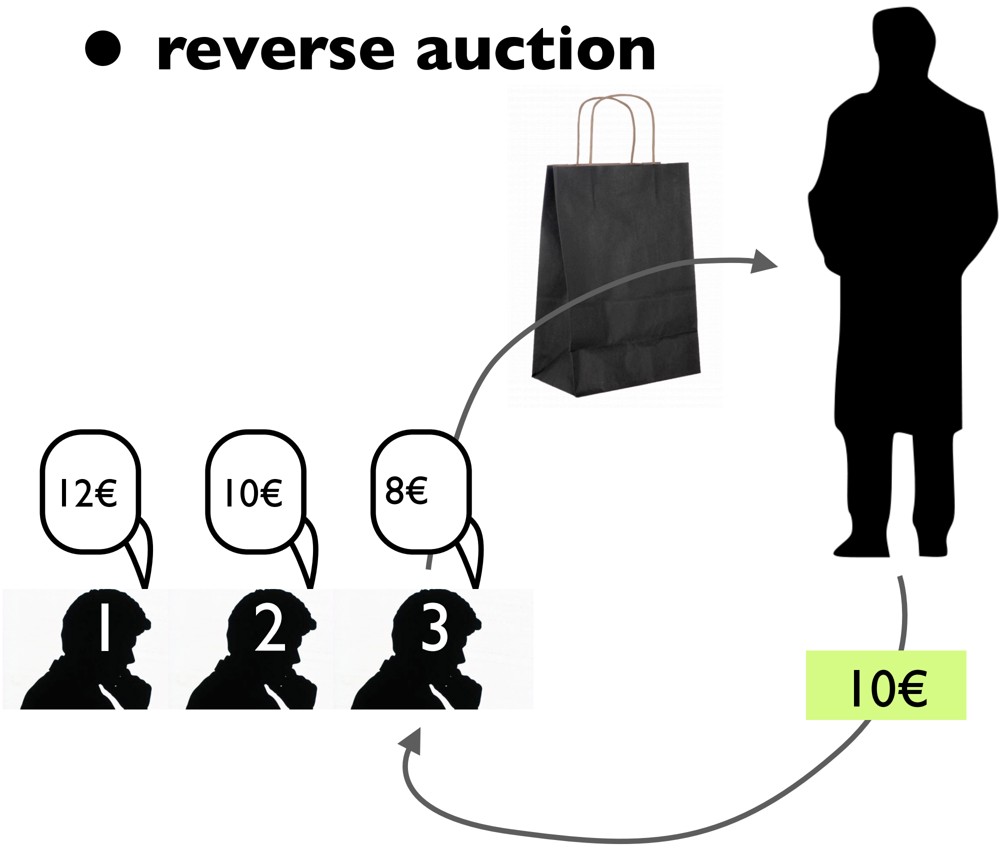
- example : 2nd price auction
- f(b) maximizes social welfare **si** f resists to manipulations
- indeed the winner k pays max b_{-k}

Clarke's pivot rule

- being VCG forces f and (p_i). But what is the good choice for functions (h_i) ?
- *desirable properties :*
- players have always a non-negative utility
  [individually rational]
  $\forall v\ \forall i: t_i(f(t))-p_i(t)\geq 0$
- we don't pay players
  [no positif transfer]
  $\forall v\ \forall i: p_i(t)\geq 0$
- **Clarke's pivot rule**
  $h_i(t_{-i}):=\max_{c\in A} \sum_{j\neq i} t_j(c)$
- with this rule the utility becomes
  $u_i=\sum_j t_j(f(t)) - \max_{c\in A} \sum_{j\neq i} t_j(c)$
- social welfare
- social welfare without player i
- u_i = change in social welfare caused by the participation of player i
  "the paiements make each player internalize the externalities that he causes"
- this rule has the desired properties as long as (t_i) are non negative.

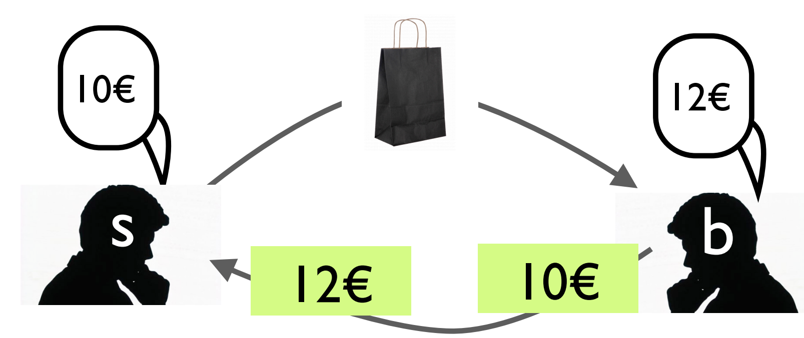
examples

- **reverse auction**



- sellers announce a price
- The buyer chooses the cheapest offer but pays him the second smallest price.

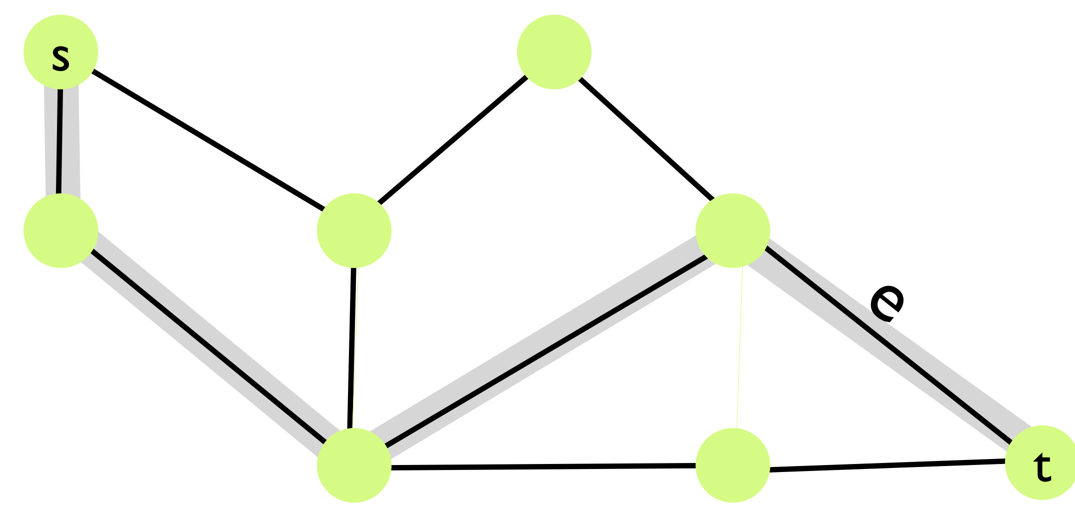
- **bilateral market**
- A single good
- for the seller it is worth $0 \leq t_s \leq 1$
  for the buyer it is worth $0 \leq t_b \leq 1$
- A={deal, no deal}
- $f(t_s, t_b)$="deal" iff $t_s \leq t_b$
- in this case $p_s = t_b$, $p_b = t_s$



- there is a loss during the transfers, but this unavoidable if we want to maximize social welfare and avoid strategic manipulations

Buy a path

- Let G(V,E) be a 2-connected graph
- Let A be the set of s-t-paths
- player e has cost c_e for using edge e. Its utility is 0 if is not used and $-c_e$ otherwise.



exercise:
determine the VCG mechanism +Clarke's rule for this game

Can we bound the payments with the length of the shortest path?

s

Buy a path

- Let G(V,E) be a 2-connected graph
- Let A be the set of s-t-paths
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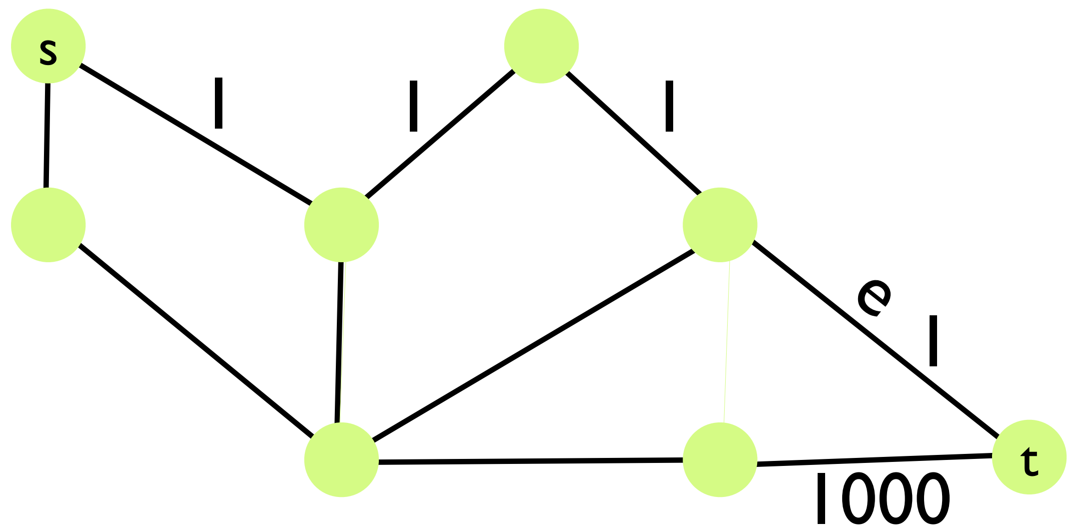

- optimize social welfare = choose shortest path
- Clarke's rule: pay to e the difference distance(s,t,G)-distance(s,t,G\e).
- Edge e will be charged at least 1000, while shortest path is only 4. → there is a need for mechanisms that do not overcharge too much.

Problem of computing social optimum

- Arrow's impossibility theorem

↓

- introduction of payments
  problem of private information

↓

- mechanism VCG

- **Theorem** [Roberts 1979] For a surjectif fucntion f on A, with |A|≥3, if f resists to manipulations then f is essentially VCG (might be a weighted variant called *affine maximisor*)

- But there is another problem. VCG requires to compute the social optimum.
  What shall we do if this is an NP-hard problem ?

↓

- We need a mechanisme approximating the optimum.
  What properties can we maintain then and how ?

- A good way is to introduce alea in the mechanism.

Summary

- We saw that there is no social choice function with resists to manipulations and is without dictator. [Gibbard-Satterwaite]
- We saw that after introducing payments, the unique mechanisms which resist to manipulations are essentially VCG.
- We saw that it is not enough to relax the optimality of the social choice in order to resist to manipulations.