On Price Induced Minmax Matchings

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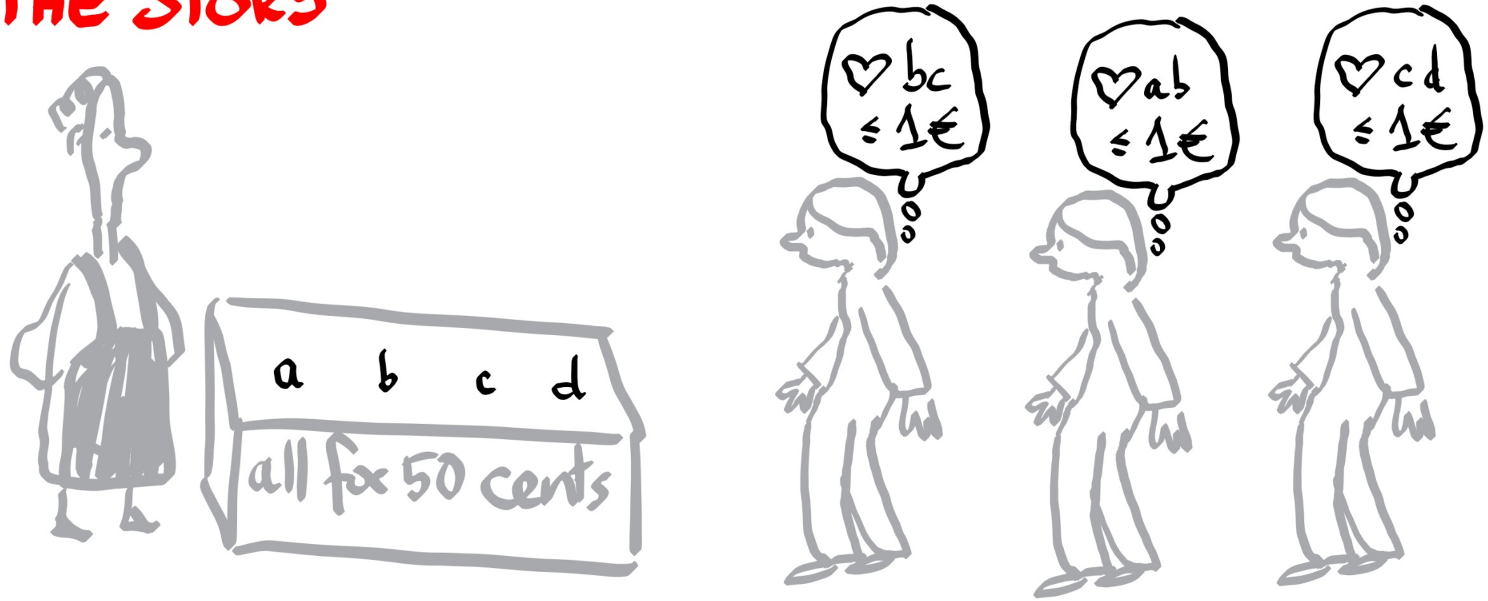
Workshop Pricing Algorithms

— Talence – Oct 2023 —

THE STORY

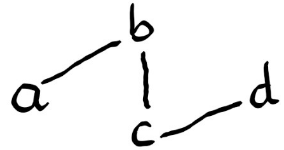


Given graph $G=(V,E)$

vertices = items

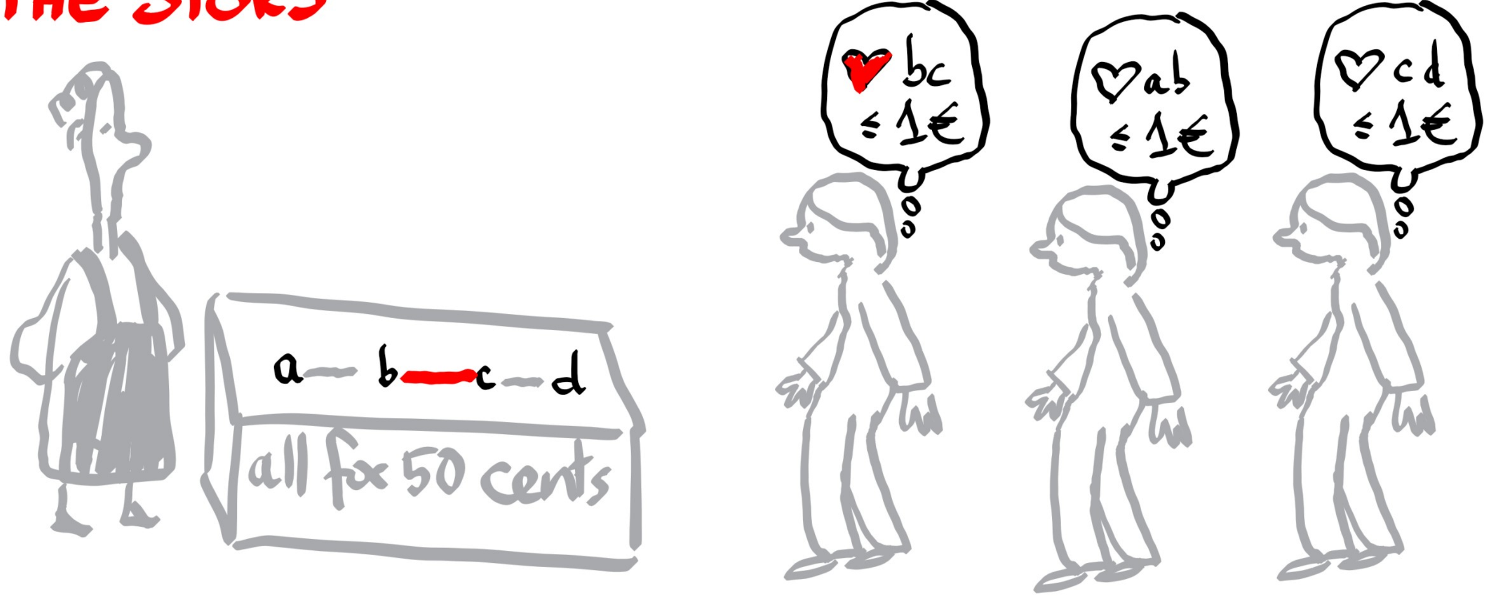
edges = buyers interested in exactly 2 items

adversarial arrival order



Objective size of minimal maximal matching

THE STORY



Given graph $G = (V, E)$

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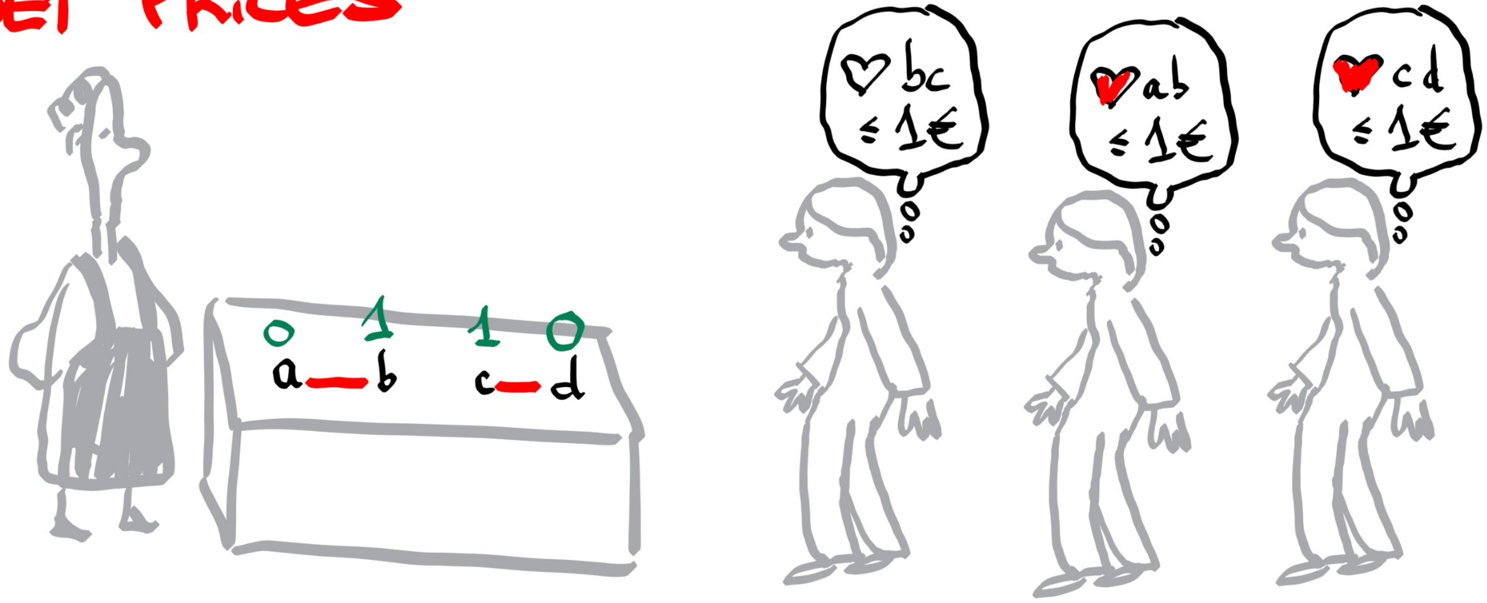
Objective size of minimal maximal matching

Compare with size of maximum matching

Ratio $\geq \frac{1}{2}$ a — b — c — d (minmax) versus a — b — c — d (maximum)

How can we do better?

SET PRICES



Given graph $G = (V, E)$

vertices = items

edges = buyers interested in exactly 2 items

adversarial arrival order

Set prices $p : V \to \mathbb{Q}^+$

expensive edges get evicted $E_p := \{uv \mid uv \in E,\ p(u) + p(v) \leq 1\}$

Objective size of minimal maximal matching in (V, E_p)

Compare with size of maximum matching in (V, E)

Ratio was known $\in [\frac{1}{2}, \frac{2}{3}]$ 0.5 $0.\overline{6}$ our contribution $\in (\frac{1}{2}, \frac{3}{5}]$ 0.5 0.6

Related work

Correa et al 2022

- every item i is available k_i times
- every buyer i has a valuation function v_i
  monotone

  $v_i : 2^{\text{items}} \to \mathbb{R}^+$ chosen from a distribution F_i

- d = size of largest sets buyers are interested in

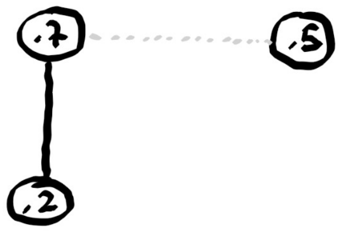
  $\forall i\ \forall A \subseteq B$ with $|A| = d$ we have $v_i(A) = v_i(B)$

- adversarial arrival order of buyers
  will buy a set A among the still available items
  maximizing
  $$v_i(A) - \sum_{j \in A} \text{price}(j)$$
- there is a price vector acheiving
  competitive ratio $d+1$

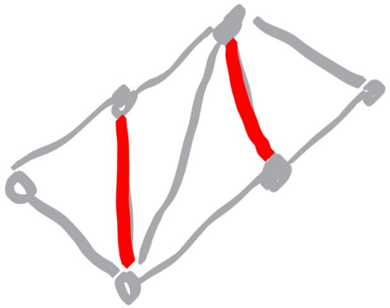
  and this is best possible
  (d items, 2 buyers)

Chvátal, Hammer 1977

- many characterizations of Threshold graphs
- these are graphs $G = (V, E)$ s.t.
  $\exists$ prices $p : V \to [0,1]$
  $uv \in E$ iff $p(u) + p(v) \leq 1$



What do we know about matchings



$G = (V, E)$

matching $M \subseteq E$

$\forall e, e' \in M$: endpoints of e, e' are disjoint

largest

maximum matching can be computed in polynomial time

ratio $\geq \frac{1}{2}$

maximal matching no edge can added

smallest

minmax matching NP-hard

hard to approximate with factor better than $1/2$ under UGC

[WAOA 2021]

KEEPABLE EDGE SETS

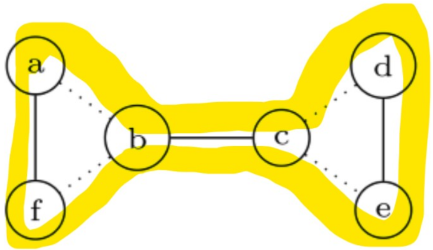
For which sets $S \subseteq E$ do we have $\exists \rho : E_\rho = S$?

What edge set $S \subseteq E$ do we want to keep?

KEEPABLE EDGE SETS

For which sets $S \subseteq E$ do we have $\exists p : E_p = S$?

example from Correa et al [IPCO22]



We would like to keep only the edges $S = \{af, bc, de\}$
But this is impossible

By summing up the prices along the S-alternating cycle
depending how we pair successive vertices
we get > 4 and ≤ 4.

(1) S is keepable iff there is no S-alternating cycle

constructive proof $S \longrightarrow$ prices p

or $\longrightarrow$ S-alternating cycle

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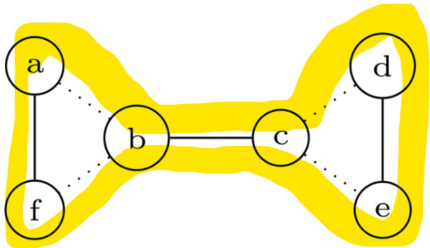

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constructive proof $S \longrightarrow$ prices p
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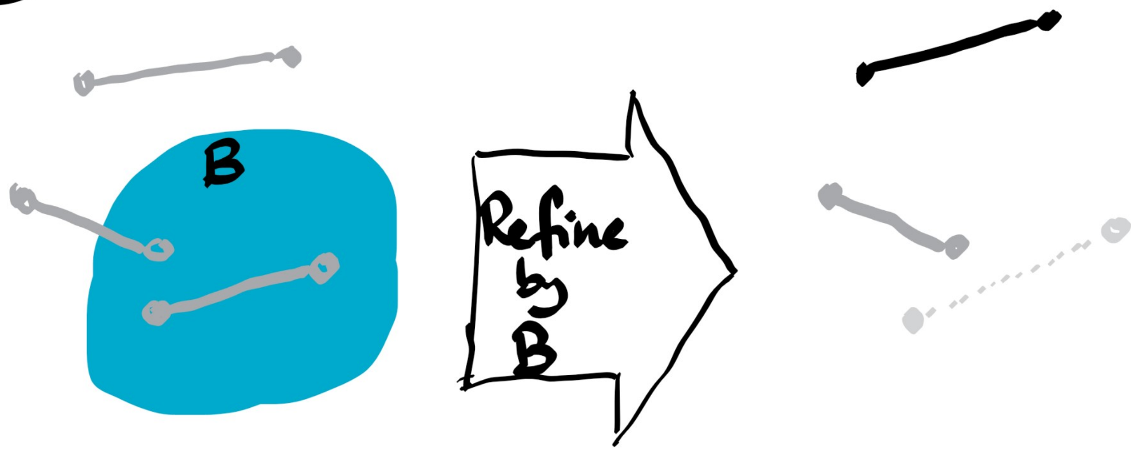
iterative procedure
looks for a vertex v either only adjacent to edges in S
or only adjacent to edges not in S
sets a price to v
removes v

KEEPABLE EDGE SETS

(2) S is keepable if and only if it is the result of a sequence of refinements.

Initially all edges are temporarily kept

Repeatedly refine by a vertex set B:



temporaly kept edges in B are evicted

temporaly kept edges outside of B are definitly kept.

This characterization is well known

Threshold graphs — Chvátal, Hammer 1977

Upper Bound: Ratio $\leq \frac{3}{5}$

We use a $\{0,1\}$-linear program and a solver to show that on the Petersen graph no minmax matching of size 4 can be reached.

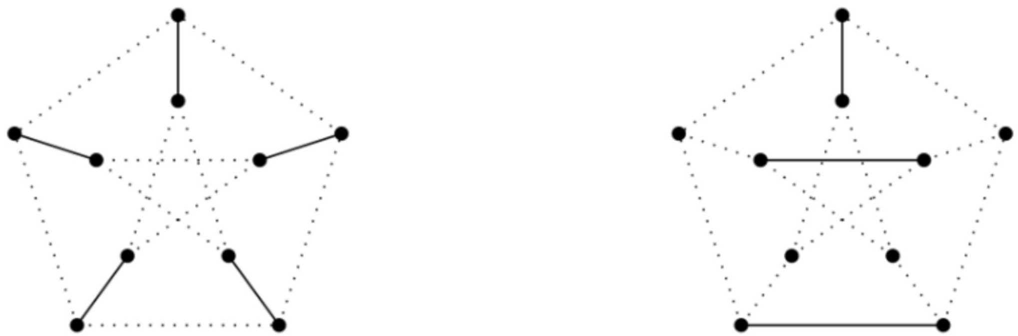


Figure 4: A maximum and a minmax matching in the Petersen graph.

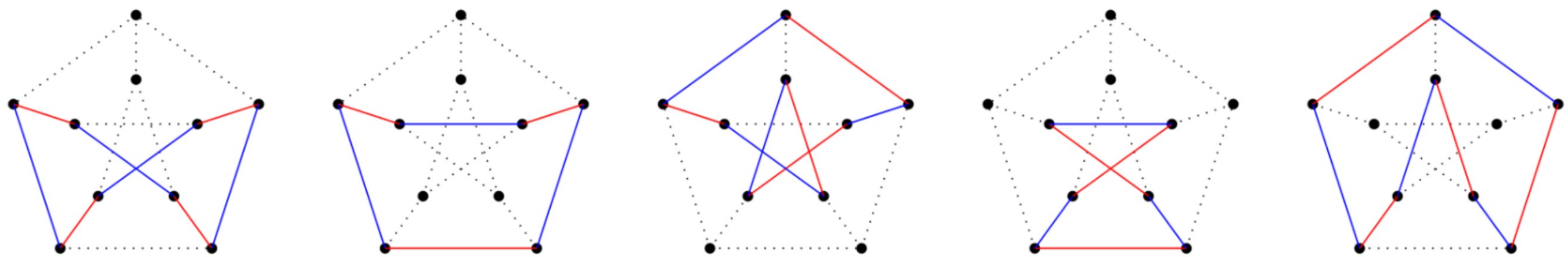


Figure 5: All 5 even length cycles up to rotations.

LOWER BOUND : Ratio > $\frac{1}{2}$

Initially all edges are temporarily kept

M^* = maximum matching (wlog assume perfect matching)

Repeat

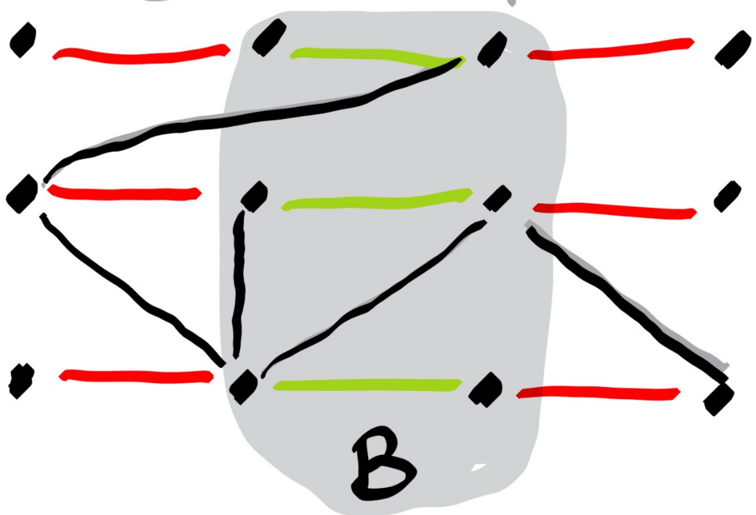
$M \leftarrow$ minmax matching (not polynomial time)

If $|M| > |M^*|/2$ then we are done

Else $M \triangle M^*$ forms alternating paths of length 3

$B \leftarrow$ all vertices matched by M

Refine by B (evicts all edges in B
no edge becomes definitely kept
by maximality of M)



Thank you


You don't know how to buy good edges at the market. Even after one hour, they are not cooked